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GENERALIZED LATITUDE AND LONGITUDE IN A GENERAL RIEMANNIAN SPACE, WITH A SPECIALIZATION FOR HOTINE'S (ω,ϕ,N) COORDINATE SYSTEM

Georges Blaha

Nova University Oceanographic Center
8000 North Ocean Drive
Dania, Florida 33004

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The present study seeks to find out whether Hotine's coordinate system (ω, ϕ, N) could be admissible in a class of general (curved) Riemannian spaces. The adopted approach postulates generalized quantities ω and ϕ to be coordinates and then examines, via commutators, in what kind of space this may hold true; the third coordinate remains N, a differentiable scalar function of position, which is assumed to be admissible in any space. The commutators lead to six conditions for six independent components of the covariant Riemann-Christoffel tensor. In the case of the original Hotine's coordinates ω and ϕ, it is shown that these components are

identically zero, and, therefore, that the space must be flat. This implies that all orders of covariant derivatives of A_r, B_r, C_r must be zero, where the orthonormal triad A, B, C represents Hotine's base vectors. Besides disproving the admissibility of the (ω, ϕ, N) coordinate system in any but the flat space, the analysis cross-validates a number of equations from [Zund, 1990], applicable to the flat space, and presents several new relations that do not have equivalents in [Hotine, 1969] or [Zund, 1990].

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1. INTRODUCTION

The present study is concerned with the construction of Hotine's (ω, ϕ, N) coordinate system as used in the three-dimensional flat space, and, especially, with the question as to whether this construction may or may not be generalized to a three-dimensional curved Riemannian space. In [Hotine, 1969], henceforth abbreviated as [H], the (ω, ϕ, N) coordinate system is built on the notion of N-surfaces, each defined by a certain constant value of the coordinate N, which can be identified with equipotential surfaces of the earth's gravity field. Statements found in §1-19, §15-2, and §17-33 of [H] imply that N is a scalar invariant (independent of the choice of a coordinate system), that it is a single-valued, continuous, and differentiable function of position throughout a region of space considered, and that within this region, adjacent normals to any given N-surface do not intersect. When, in the following, we use the attribute "well-behaved" for the scalar N, these are the qualities we will have in mind.

As is explained in §12-1 through §12-10 of [H], the other two space coordinates in Hotine's system are the independent scalars ω (longitude) and ϕ (latitude), which are functions of the normal to a given N-surface at a given point. A well-behaved scalar N in a region considered implies that there are no singular points in the (ω, ϕ, N) coordinate system, i.e., that there are no distinct points having identical coordinates Such a well-behaved scalar N is sometimes interpreted as giving rise to non-intersecting, convex surfaces in that region.

With regard to all points on the earth's surface, some investigators suggest the adoption of a coordinate system $(\omega, \phi, |\text{grad } N|)$, the usefulness of which they justify by the existence of singular points in the (ω, ϕ, N) system. However, since we are not concerned with the extent of regions where the system (ω, ϕ, N) is suitable, we do not address this proposition. The derivations in the present study, focusing on the theoretical question with regard to a potential admissibility of the system (ω, ϕ, N) for a curved space, pertain to an infinitesimal neighborhood of a given point P. In this neighborhood, N is assumed to be well-behaved. If the feasibility of the (ω, ϕ, N) system in this neighborhood under the stipulation of a curved space is established, a following step would be to validate the system for finite regions. If the system is not feasible in this neighborhood, the outcome of the analysis is declared negative and the investigation terminates.

The feasibility study of the extension of the (ω, ϕ, N) coordinate system to a curved space is facilitated by the fact that the scalar N can be accepted as the third space coordinate whether the space is flat or curved. This stems from the tensor equation

$$N_{rs} = N_{sr} ,$$

valid, under the usual assumptions about N, at any point of a region of space considered (here it is needed only within an infinitesimal neighborhood of P). The above equality stems from the first and second covariant derivatives of a scalar (and the first covariant derivatives of a vector) being unaffected by the curvature of the space. Since this type of formula will be the basis for acceptability of a given quantity as a coordinate, the scalar N (which has already qualified) will not be subject to further scrutiny. On the other hand, the question arises as to whether ω and ϕ could under certain circumstances be admissible as the other two space coordinates in a curved space, similar in this respect to N, or whether they are admissible as such coordinates strictly in the flat space. This question will be answered as a by-product of the analysis concerned with the generalization of ω and ϕ to G and H, which are called respectively the generalized longitude and latitude.

The approach we choose is admittedly unusual. In practice, one usually designs suitable coordinates in a two-, three-, or higher-dimensional space whose characteristics are known. However, here we postulate that the quantities G and H are admissible as coordinates, and examine what this postulate entails for a three-dimensional space in terms of the covariant Riemann-Christoffel tensor and its spatial derivatives. Subsequently, we specialize G and H to ω and ϕ, and examine whether the covariant Riemann-Christoffel tensor and all of its spatial derivatives are required to be zero. In the affirmative, we conclude that ω and ϕ can exist as coordinates only in the flat space. This, in fact, would represent the negative outcome of the feasibility study of extending the (ω, ϕ, N) coordinate system to a curved space.

As we have indicated, in Chapter 12 Hotine [1969] constructs his (ω, ϕ, N) coordinate system for use in the flat space. However, he considers certain equations related to the construction of his coordinate system as being valid only in the flat space, whereas it will be shown later in this Introduction that such equations could be valid in a curved space as well. This, in fact, has led above to the challenging question as to whether the coordinates ω and ϕ could, by chance, be admissible in some class of curved spaces, and hence to the task referred to as "feasibility study". Another challenging task is to approach the problem via generalized latitude and longitude, task that may be of interest in its own right.

Although a lower-dimensional analysis is not likely to provide reliable guidance for the present feasibility study, such analysis was undertaken for the sake of interest. The role of the flat space was taken by a plane, the role of an N-surface was taken by an N-curve, and the role of the coordinates ω and ϕ was taken by the coordinate ψ. In analogy to Hotine's system, the coordinate ψ is the angle between a "base" vector C considered at a given point P on the N-curve, and the outward-drawn normal to this curve at P. (Since N is assumed to be well-behaved, there are no singular points in the ψ,N coordinates.) Upon using a lower-dimensional analogue of the analysis carried out in the subsequent chapters, the condition for ψ being admissible as a coordinate in the neighborhood of P turns out to be $R_{\alpha\beta\gamma\delta}\ell^{\alpha} j^{\beta} \ell^{\gamma} j^{\delta}=0$, where $R_{\alpha\beta\gamma\delta}$ is the covariant Riemann-Christoffel tensor in two dimensions, ℓ is the unit tangent to the N-curve, and j is the normal to the N-curve, all associated with P. By virtue of §5-20 in [H], this condition translates into K=0, where K is the Gaussian curvature. However, K=0 allows for developable surfaces (e.g., a cylinder). Although such surfaces are not flat and cannot be expressed by Cartesian coordinates, they do not exclude ψ as a coordinate. Thus, a lower-dimensional analysis has not eliminated the possibility that a class of curved spaces might exist, where ω and ϕ would be acceptable as coordinates (N being acceptable by definition).

We remark that the approach chosen for the present study relies heavily on the covariant Riemann-Christoffel tensor. In §5-5 of [H], this fourth-order tensor is called the "covariant form" of the Riemann-Christoffel tensor; the latter is presented in §5-3 [ibid]. For the sake of brevity, the Riemann-Christoffel tensor will be called here the R-tensor, and the covariant Riemann-Christoffel tensor will be called the covariant R-tensor. We illustrate the importance and usefulness of these tensors in Section A.2 of Appendix A.

Throughout the analysis, Hotine's notation for these and other tensors, indices, etc., will be adhered to.

In order to explain, along general lines, the link between the coordinates ω and ϕ of the (ω, ϕ, N) system on one hand and the covariant R-tensor on the other, we recall that according to Chapter 12 of [H], ω and ϕ are determined with the aid of the orthonormal "base" vectors A, B, and C. If ω and ϕ are to be coordinates not only at a given point, denoted P, but at neighboring points as well, a question arises with regard to the parallel transport, or possibly some other kind of transport, of A, B, and C from the point P. In [H] the base vectors emanate from the known point called "origin", but by virtue of the parallel transport in the flat space, they could equally well emanate from P or any other point. Since here we work with the neighborhood of P, it is expedient to consider the base vectors as emanating from P. The question of transporting A, B, and C is related to the quality of the space in the neighborhood of P (along the N-surface at P as well as along neighboring N-surfaces), which, in turn, is related to the covariant R-tensor in the neighborhood of P, or, equivalently, to the covariant R-tensor and its spatial derivatives at P. If the space is flat, this tensor is zero at all points of the space, or, equivalently, this tensor and all of its spatial derivatives are zero at P.

The statement of equivalence between the covariant R-tensor being zero at different points in the space, and this tensor and all its spatial derivatives being zero at P, is rooted in the consideration that this and other tensors can be expressed at various locations via the Taylor-series expansion from P. Such an expansion has an important role to play in the present study, and is treated in sufficient detail in Appendix A. Since an expression for tensor components at P' based on tensor components at P pertains to distinct points by definition, the equality sign in the Taylor series does *not* indicate a tensor equation, but merely a collection of equalities for individual components. However, that such a collection of equalities does not represent a tensor equation does not detract from its usefulness.

We illustrate the above discussion with a simple example in a two-dimensional flat space, i.e., a plane. The coordinates in this plane are chosen to be r (distance from the origin) and θ (angle from a given axis); thus, the coordinate system is represented by $(u^{\alpha}) \equiv (r, \theta)$. The metric in these coordinates is $ds^2 = dr^2 + r^2 d\theta^2$, implying the metric tensor in matrix notation:

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} ,$$

which is assumed to be valid at the point P. Since for the point P' we have $r'^2=(r+\Delta r)^2$, where $\Delta r=r'-r$, it follows that

$$[a'_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2r\Delta r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Delta r^2 \end{bmatrix} .$$

But this is precisely the equality one obtains when proceeding via the Taylor series as outlined in Section A.1 of Appendix A, except that here the coordinate system is not locally Cartesian at P (i.e., $[a_{\alpha\beta}]\neq$identity matrix). This equality is derived in detail in Appendix B.

We now return to the coordinates ω and ϕ of the (ω,ϕ,N) system, and show that certain relations used in [H] do not apply strictly in the flat space, which leaves open the question of additional possibilities. We start with the argument that if one were working in a general Riemannian space, one could still use Hotine's orthonormal base vectors A, B, and C, and consider them in a locally Cartesian coordinate system at P. This would be *sufficient* to produce the three tensor equations implicit in the development of Chapter 12 in [H]:

$$A_{rs} = B_{rs} = C_{rs} = 0 , \qquad (1)$$

where A_{rs} is obtained by the covariant differentiation of A_r, etc., and where the covariant components of A, B, and C in the above-mentioned local system are $A_r=(1,0,0)$, $B_r=(0,1,0)$, and $C_r=(0,0,1)$. Since the tensor equations represented by (1) stem from the definition of a locally Cartesian system, which is admissible in the flat space as well as in curved spaces, they, alone, are inconsequential with regard to the flatness of the space. Accordingly, their use in the derivations leading to Hotine's equations (12.046,047) for ω_r and ϕ_r does not restrict these results to the flat space.

It is thus apparent that the same formulas giving ω_r and ϕ_r would have been obtained even if the space were known to be curved. An identical statement can be made also with regard to equations (12.014-016) in [H] giving λ_{rs}, μ_{rs}, and ν_{rs}, where λ, μ, and ν form an orthonormal triad, with λ and μ being tangent to the N-surface and ν being an outward-pointing normal. (These and all the other tensor equations in this study are considered at P, where they are valid in any coordinates.) Since much of the development in §12-88 through §12-94 is based on the over-restrictive assumption that the formulas for λ_{rs}, μ_{rs}, and ν_{rs} apply only in the flat space, Chapter 12 of [H] is incapable of formulating sufficient conditions for N-surfaces to be embedded in the flat space. We will return to considering ω and ϕ in the flat space in conjunction with the integrability conditions, and first present a short discussion pertaining to the flat-space characteristics in terms of Hotine's vectors A, B, and C.

The above space ambiguity is linked to the limitations of (1), which merely states that the system associated with A, B, and C is Cartesian to a first order, i.e., Cartesian at P and in its immediate neighborhood, and bears no relationship to the curvature of the space. (As is indicated in §5-8 of [H] and described in Section A.1 of Appendix A, if, in the Taylor-series expansion of the metric tensor from P, the term linear in coordinate differences is missing, the coordinate system is called Cartesian to a first order, if also the quadratic term is missing, the coordinate system is called Cartesian to a second order; etc.) The flatness of the entire space is expressed through further tensor equations in addition to (1), obtained by covariant differentiation as

$$A_{rst} = B_{rst} = C_{rst} = 0 \ , \tag{2}$$

$$A_{rstu} = B_{rstu} = C_{rstu} = 0 \ , \tag{3}$$

etc. Section A.3 of Appendix A shows that equation (2) is equivalent to

$$k_{rst} = k_{rts} \ , \tag{4}$$

where k is a general vector in space. The derivation of the Mainardi-Codazzi equations in (6.21) and later in (8.23) of [H] makes use of the above formula (4) specialized for ν. But even if (4) were not specialized, the local system could be considered Cartesian only to a second order if (3), etc., did not hold true. The use of such a Cartesian system would be confined to a relatively small neighborhood of P.

An arbitrary differentiable scalar function of position, denoted F, is admissible as a coordinate in a given space if it fulfills three integrability conditions that Zund [1990] calls "commutators". The commutators of F in a general space can be derived upon using the symmetry of F_{rs} in r and s, and are, indeed, equivalent to the condition $F_{rs}=F_{sr}$. In Chapter 12, Hotine attempted to formulate six flat-space integrability conditions for his system, where the scalar functions of position ω and ϕ were intended as the first two coordinates. He considered two of these conditions to be the Mainardi-Codazzi equations. He formulated three further conditions, namely (12.138-140), and aggregated them into the surface tensor equation (12.144). In §12-92, he concluded that the sixth condition is represented by the symmetry of N_{rs}.

However, the three conditions just mentioned were derived upon assuming that the formulas for λ_{rs}, μ_{rs}, and ν_{rs} are valid strictly in the flat space, whereas we have seen that they involve only (1) but not (2), (3), etc., and could thus be valid in other spaces as well. This finding is consistent with the fact that the curvature of the space does not affect the first covariant derivatives of vectors, here λ_{rs}, μ_{rs}, and ν_{rs}. A similar negative comment can be made about the sixth condition, which does not involve any of (1), (2), (3), etc. In particular, since the ordinary second-order partial derivatives commute and the Christoffel symbols are symmetric in the lower indices regardless of the kind of the space and of the kind of coordinates in use, it follows for any space where N is a scalar invariant (assumed well-behaved as usual) that

$$N_{rs} = N_{sr} , \tag{5}$$

which has been encountered earlier. Clearly, the flat space has no special distinction in this regard. Finally, the remaining two conditions represented by the Mainardi-Codazzi equations are weaker than (4), or, equivalently, weaker than (2), which, in any event, allows for spaces other than the flat space.

The main outcome of the introductory discussion can be summarized as follows. Chapter 12 of [H] is inconclusive as to the conditions under which the coordinates ω and ϕ of the (ω,ϕ,N) system are restricted to the flat space. Thus, a theoretical possibility of ω and ϕ being admissible as the first two coordinates in a general (curved) space will be investigated. The vehicle in carrying out this task will be the generalized longitude and latitude, G and H. The theoretical possibility just mentioned will be precluded if it is proved that the covariant R-tensor must be identically zero in a space where ω and ϕ are stipulated to be coordinates. Such a proof, or a counter-proof, is a major goal of the present study.

2. OUTLINE OF A "REVERSED" APPROACH

2.1 Role of the Covariant Riemann-Christoffel Tensor

In order to assess the admissibility of ω and ϕ as coordinates in the flat space and to answer the questions raised in the Introduction, we undertake a "reversed" approach, in which the space will be required to accommodate two functions such as ω and ϕ postulated to be coordinates. The third coordinate will always be N, a continuous differentiable scalar function of position, which is defined to be an invariant in any space. Accordingly, (5) holds true by definition. The designation "any space" or "general space" pertains here to a Riemannian space with general curvature properties, i.e., the flat space or curved spaces (in three dimensions). The two quantities postulated as coordinates in some spaces are denoted G and H, and are called "parameters". They are considered to be quite general differentiable functions satisfying the commutators, or, equivalently, the conditions

$$G_{rs} = G_{sr} , \qquad H_{rs} = H_{sr} , \tag{6a,b}$$

in a class of spaces. In such spaces they will be proper scalar invariants and will be admissible as coordinates. We postulate that (6a,b) hold true and proceed to find out all we can about such a class of spaces.

The specialization of G and H for Hotine's coordinates ω and ϕ will take place in the final stage of the analysis. For the most part, either of the commutators in (6a,b) is represented by

$$F_{rs} = F_{sr} , \tag{7}$$

where F is an almost arbitrary differentiable scalar function of position. In the next chapter, the commutators will be derived from this symmetry condition, which also supplies ΔF, the Laplacian of F, with little additional effort. The main thrust of the present approach is the stipulation that the commutators embody conditions on the space, in particular, on the covariant Riemann-Christoffel tensor (covariant R-tensor), rather than on the parameters G and H. This tensor can be used to express the space in which G and H are admissible as coordinates via the metric tensor constructed from the point P outwards. Such a procedure begins upon stipulating a locally Cartesian coordinate system at P, denoted $\{x^r\}$, r=1,2,3. By construction, the system $\{x^r\}$ is Cartesian to a first order, but it could also be Cartesian to a second, a third, etc., orders, or it could be globally Cartesian, i.e., Cartesian to all orders.

The properties of $\{x^r\}$ are reflected by the metric tensor expressed at arbitrary points. This tensor can be found from the Taylor-series expansion of g_{rs}, the metric tensor at P, the matrix form of which at P is $[g_{rs}]=I$. The expansion proceeds in terms of the Christoffel symbols (C-symbols) and their partial derivatives with respect to the coordinates $\{x^r\}$. The C-symbols are zero at P due to the definition of the system $\{x^r\}$. Accordingly, the linear term in the expansion of the metric tensor is missing, which is precisely what makes the system "Cartesian to a first order". The quadratic term contains the first-order partial derivatives of the C-symbols, the cubic term contains their second-order partial derivatives, the next terms contains their third-order partial derivatives together with their first-order partial derivatives, and so on. If all orders of partial derivatives of the C-symbols are zero, the system is globally Cartesian and vice versa. These derivatives can be determined from the components of the covariant R-tensor and their partial derivatives, but are not unique. In the absence of conflicting information we set them to zero as a part of the strategy in constructing the system $\{x^r\}$. This strategy has the advantage of seizing every opportunity to make the system $\{x^r\}$ globally Cartesian. If this actually occurs, one concludes that the space must be flat.

In general, the C-symbols are linked to the covariant R-tensor via the Riemann-Christoffel tensor (R-tensor). In §5-6 of [H], the R-tensor is related to the first-order partial derivatives of the C-symbols in a locally Cartesian coordinate system that has been identified here by $\{x^r\}$, and at a point that has been identified here by P. We elaborate and expand this subject in Section A.2 of Appendix A. In particular, in working with the system $\{x^r\}$ and considering all the relations at P, we link both versions of the R-tensor and their partial derivatives to each other, as well as to the partial derivatives of the C-symbols, which, in turn, we link to the Taylor-series expansion of the metric tensor. This approach shows how the covariant R-tensor can help to express the space in a concrete manner.

In three dimensions the covariant R-tensor contains (except for possible sign differences) only six distinct components. We thus need six independent commutators featuring these components as unknowns. In general, the commutators will yield the six components in terms of the parameters G and H. If these components all turn out to be zero at P, all components of the covariant R-tensor must be zero and all of the first-order partial derivatives of the C-symbols are admissible to be zero. In terms of the Taylor-series expansion of the metric tensor, this indicates that $\{x^r\}$ is Cartesian to a second order, and could possibly be Cartesian to a higher order. If, in addition, the first-order partial derivatives of six independent components (and thereby of all components) of the covariant R-tensor are likewise zero, similar reasoning indicates that $\{x^r\}$ is Cartesian to at least a third order. If they are not zero, the system is Cartesian only to a second order.

Accordingly, the necessary and sufficient conditions for the system $\{x^r\}$ to be globally Cartesian is contained in equation (A.19) of Appendix A, namely

$$R_{uijk} \equiv \text{constant} = 0 \ , \tag{8}$$

which implies that the covariant R-tensor and *all* of its spatial derivatives at P are zero. The indices in (8) may be interpreted to represent either all components, or six independent components; the two interpretations are mathematically equivalent. Due to the fact that a globally Cartesian coordinate system may exist by definition only in the flat space, (8) represents the necessary and sufficient conditions for the space to be flat.

In summary, the "reversed" approach pursued herein uses the commutators to express six independent components of the covariant R-tensor and their partial derivatives at P via the parameters G and H. This leads to restrictions on the space where G and H are admissible as coordinates. If (8) holds true, the space is necessarily flat and $\{x^r\}$ represents a Cartesian coordinate system in this space. If (8) does not hold true, the space is curved and $\{x^r\}$ is Cartesian only to a certain order. The actual values of the components of the covariant R-tensor and of their partial derivatives at P can serve to express the metric tensor, in the coordinate system $\{x^r\}$, at arbitrary points in this space.

2.2 General Formulation of the Parameters G and H

In order to allow the analysis to proceed along the most general lines possible, we impose only few broad conditions on the parameters G and H. The latter are considered at the point P, where we have introduced a locally

Cartesian coordinate system $\{x^r\}$ associated with the orthonormal triad A, B, C. At the same point, we define another orthonormal triad λ, μ, ν, where ν is perpendicular to the pertinent N-surface. Since the latter is defined via N=constant (N being one coordinate of the system) regardless of whether the space is flat or curved, the basic gradient relation, namely

$$N_r = n\nu_r \ , \tag{9}$$

is a tensor equation at P, valid in any space and in any coordinates (such as the locally Cartesian coordinates, some general underlying coordinates, etc.). The scalar n in (9) is the magnitude of the gradient vector N_r. The above-mentioned broad conditions on the parameters are stipulated to be of any kind that allows the triad λ, μ, ν to be uniquely related to the triad A, B, C via G and H. To illustrate one such possibility, we project the known vector ν onto a plane formed by two of the three axes A, B, C, denoting the unit vector in this direction by ν'. The oriented angle between one of these two axes and ν' can constitute one parameter, and the oriented angle between ν' and ν can constitute the other. Since such a construction does not involve (second-order) covariant differentiation, it can be used unaltered whether the space is flat or curved.

Conversely, G and H can uniquely determine the triad λ, μ, ν. In the above illustration, ν is constructed by back-tracking the definition of G and H. These two parameters can then also express the plane perpendicular to ν at P, i.e., the plane tangent to the N-surface and containing λ and μ. As soon as one of λ and μ is oriented with respect to a known direction in this plane, both vectors λ and μ are uniquely expressed by the two parameters. The coordinates ω and ϕ represent a special case in this illustration. In particular, the two axes forming the plane into which ν is projected are A and B, and the axis used for the orientation of ν' is A. The first parameter, ω, is the oriented angle between A and ν', and the second parameter, ϕ, is the oriented angle between ν' and ν. With regard to the "known direction" in the plane tangent to the N-surface, it is now materialized by the intersection of the tangent plane with the plane defined by ν and C. The vector to be oriented with respect to this direction is λ, and the orientation angle is zero. In Chapter 12 of [H], the definition of ω and ϕ based on λ, μ, ν can be provided in several ways from (12.008), such as given by (12.003-005), and the unique determination of λ, μ, ν from ω and ϕ is provided by equations (12.008) themselves. Here again, the curvature of the space does not modify these procedures because it only affects second- and higher-order covariant derivatives of vectors, or third- and higher-order covariant derivatives of scalars.

In general, the unique determination of λ, μ, and ν by the parameters G and H is represented by

$$\lambda_r = f_1 A_r + f_2 B_r + f_3 C_r \ , \tag{10a}$$

$$\mu_r = g_1 A_r + g_2 B_r + g_3 C_r \ , \tag{10b}$$

$$\nu_r = h_1 A_r + h_2 B_r + h_3 C_r \ , \tag{10c}$$

where $f_i \equiv f_i(G,H)$, $g_i \equiv g_i(G,H)$, $h_i \equiv h_i(G,H)$; $i=1,2,3$. In compact notation, this system of tensor equations, where the index r can also be raised, is written as

$$\begin{bmatrix} \lambda_r \\ \mu_r \\ \nu_r \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{bmatrix} \begin{bmatrix} A_r \\ B_r \\ C_r \end{bmatrix} . \tag{10'}$$

The formula (10') can be regarded as a matrix equation, where each of λ_r, μ_r, ν_r and A_r, B_r, C_r represents a row-vector of three elements. The transpose of the matrix on the left-hand side of (10') would be written as $[\lambda_r, \mu_r, \nu_r]$, where each of λ_r, μ_r, ν_r would now represent a column-vector of three elements. Thus, when the quantities such as λ_r, ... or λ^r, ... are written beneath each other they represent row-vectors; when they are written next to each other they represent column-vectors.

Since the triads λ, μ, ν, and A, B, C are orthonormal, the matrix of coefficients in (10') is orthogonal. In particular, the inverse of this matrix equals its transform, i.e., this matrix pre- or post-multiplied by its transpose yields the identity matrix. Accordingly, we are in the presence of the following six independent constraints:

$$f_1^2 + f_2^2 + f_3^2 = 1 \ , \qquad g_1^2 + g_2^2 + g_3^2 = 1 \ ,$$

$$f_1 g_1 + f_2 g_2 + f_3 g_3 = 0 \ , \qquad g_1 h_1 + g_2 h_2 + g_3 h_3 = 0 \ ,$$

$$f_1 h_1 + f_2 h_2 + f_3 h_3 = 0 \ , \qquad h_1^2 + h_2^2 + h_3^2 = 1 \ .$$

Whether the coordinate system $\{x^r\}$ is locally Cartesian (in a curved space) or globally Cartesian (in the flat space), A_r, B_r, and C_r behave as constants under the first-order covariant differentiation as implied by (1). Thus, in conjunction with (10') giving A_r, B_r, C_r in terms of λ_r, μ_r, ν_r (via the orthogonal matrix of coefficients), and with the above six constraints, (in fact, their partial derivatives), after straightforward algebraic manipulations, the covariant differentiation of λ_r, μ_r, and ν_r from (10a-c) yields

$$\lambda_{rs} = (T_{12}\mu_r + T_{13}\nu_r)G_s + (T'_{12}\mu_r + T'_{13}\nu_r)H_s \ , \tag{11a}$$

$$\mu_{rs} = (-T_{12}\lambda_r + T_{23}\nu_r)G_s + (-T'_{12}\lambda_r + T'_{23}\nu_r)H_s \ , \tag{11b}$$

$$\nu_{rs} = (-T_{13}\lambda_r - T_{23}\mu_r)G_s + (-T'_{13}\lambda_r - T'_{23}\mu_r)H_s \ , \tag{11c}$$

where $G_s \equiv \partial G/\partial x^s$, $H_s \equiv \partial H/\partial x^s$, and where

$$T_{12} = (\partial f_1/\partial G)g_1 + (\partial f_2/\partial G)g_2 + (\partial f_3/\partial G)g_3 \ , \tag{12a}$$

$$T_{13} = (\partial f_1/\partial G)h_1 + (\partial f_2/\partial G)h_2 + (\partial f_3/\partial G)h_3 \ , \tag{12b}$$

$$T_{23} = (\partial g_1/\partial G)h_1 + (\partial g_2/\partial G)h_2 + (\partial g_3/\partial G)h_3 \ ; \tag{12c}$$

$$T'_{12} = (\partial f_1/\partial H)g_1 + (\partial f_2/\partial H)g_2 + (\partial f_3/\partial H)g_3 \ , \tag{12a'}$$

$$T'_{13} = (\partial f_1/\partial H)h_1 + (\partial f_2/\partial H)h_2 + (\partial f_3/\partial H)h_3 \ , \tag{12b'}$$

$$T'_{23} = (\partial g_1/\partial H)h_1 + (\partial g_2/\partial H)h_2 + (\partial g_3/\partial H)h_3 \ . \tag{12c'}$$

The three primed quantities have the structure of their unprimed counterparts, except that H replaces G. Having formulated λ_r, μ_r, and ν_r in terms of general functions of G and H (above symbolized by f_i, g_i, and h_i, i=1,2,3), and having differentiated them covariantly, we need to introduce curvatures associated with λ, μ, and ν before proceeding further.

3. TENSOR INVARIANTS IN A GENERAL SPACE

3.1 Curvature Parameters

In following the usage by Hotine [1969] and Zund [1990], we introduce five curvature parameters of the system associated with N-surfaces, as well as three additional curvatures. The curvature parameters are k_1, k_2, t_1, γ_1, and γ_2, and the additional curvatures are σ_1, σ_2, and ε_3. These quantities correspond to the directions of the orthonormal triad λ, μ, ν. In analogy to the terminology "Hotine 3-leg" employed by Zund [1990], we refer to this triad as the "general 3-leg" (here λ and μ are not constrained to Hotine's definition of λ and μ). All of the above eight curvatures are identified as follows: k_1 and k_2 are respectively the normal curvatures of the N-surface in the λ and μ directions; $\pm t_1$ are respectively the geodesic torsions of the N-surface in these directions; γ_1 and γ_2 are respectively the curvature constituents of the normal to the N-surface in these directions; σ_1 and σ_2 are respectively the geodesic curvatures of N-surface curves in these directions; and, according to [Zund, 1990], ε_3 is a complicated expression involving the geodesic torsions of the surface curves. All of these curvatures are *scalar invariants considered* at P. They involve (single) covariant differentiation of λ_r, μ_r, and ν_r, and subsequent double contractions with some of the contravariant vectors λ^r, μ^r, and ν^r. Since no double covariant differentiation of vectors takes place, these invariants are unaffected by the curvature of the space.

Most of the formulas expressing these curvatures are adopted from Chapter 7 of [H], upon replacing the notation k, k*, t, σ, and σ^* respectively by k_1, k_2, t_1, σ_1, and σ_2, and correspondingly replacing ℓ and j by λ and μ. The formula for σ_1 follows from Hotine's equation (7.04), while that for σ_2 can be derived by similar means. The relation giving k_1 is adopted from (7.03), while that for k_2 can again be derived similarly. Finally, the formula for t_1 is adopted from (7.08). On the other hand, §12-17 of [H] yields γ_1 and γ_2, and Zund [1990] defines the contraction yielding ε_3 (below his equation 2). The contractions producing the eight curvatures are listed as follows:

$$\lambda_{rs}\mu^r\lambda^s = -\mu_{rs}\lambda^r\lambda^s = \sigma_1 \ , \tag{13a}$$

$$\lambda_{rs}\mu^r\mu^s = -\mu_{rs}\lambda^r\mu^s = \sigma_2 \ ; \tag{13b}$$

$$\lambda_{rs}\nu^r\lambda^s = -\nu_{rs}\lambda^r\lambda^s = k_1 \, , \tag{14a}$$

$$\mu_{rs}\nu^r\mu^s = -\nu_{rs}\mu^r\mu^s = k_2 \, ; \tag{14b}$$

$$\lambda_{rs}\nu^r\mu^s = -\nu_{rs}\lambda^r\mu^s = \mu_{rs}\nu^r\lambda^s = -\nu_{rs}\mu^r\lambda^s = t_1 \, ; \tag{15}$$

$$-\lambda_{rs}\nu^r\nu^s = \nu_{rs}\lambda^r\nu^s = \gamma_1 \, , \tag{16a}$$

$$-\mu_{rs}\nu^r\nu^s = \nu_{rs}\mu^r\nu^s = \gamma_2 \, ; \tag{16b}$$

$$\lambda_{rs}\mu^r\nu^s = -\mu_{rs}\lambda^r\nu^s = \varepsilon_3 \, . \tag{17}$$

The alternative expressions in (13a)-(17) displaying the opposite sign stem from equation (3.20) in [H]. All of (13a)-(17) are derived in detail in Appendix C.

The expressions for the tensors λ_{rs}, μ_{rs}, and ν_{rs} in terms of the eight curvatures are based on the following identity, with the general vector k to be substituted for by any of λ, μ, and ν:

$$k_{rs} = (k_{mn}\lambda^m\lambda^n)\lambda_r\lambda_s + (k_{mn}\lambda^m\mu^n)\lambda_r\mu_s + (k_{mn}\lambda^m\nu^n)\lambda_r\nu_s + \dots \, .$$

where the dots indicate six similar terms corresponding to the leg combinations $\mu\lambda$, $\mu\mu$, $\mu\nu$, $\nu\lambda$, $\nu\mu$, and $\nu\nu$. This identity is readily confirmed upon the contractions, in turn, with $\lambda^r\lambda^s$, $\lambda^r\mu^s$, $\lambda^r\nu^s$, ... With the aid of the scalar invariants from (13a)-(17), the above identity yields

$$\lambda_{rs} = \sigma_1\mu_r\lambda_s + \sigma_2\mu_r\mu_s + \varepsilon_3\mu_r\nu_s + k_1\nu_r\lambda_s + t_1\nu_r\mu_s - \gamma_1\nu_r\nu_s \, .$$

$$\mu_{rs} = -\sigma_1\lambda_r\lambda_s - \sigma_2\lambda_r\mu_s - \varepsilon_3\lambda_r\nu_s + t_1\nu_r\lambda_s + k_2\nu_r\mu_s - \gamma_2\nu_r\nu_s \, ,$$

$$\nu_{rs} = -k_1\lambda_r\lambda_s - t_1\lambda_r\mu_s + \gamma_1\lambda_r\nu_s - t_1\mu_r\lambda_s - k_2\mu_r\mu_s + \gamma_2\mu_r\nu_s \, ,$$

featured in equation (7) of [Zund, 1990]. This reference is henceforth abbreviated as [Z].

If these tensors are contracted with any of λ^s, or μ^s, or ν^s, they produce the first useful set of formulas below. If the index s in the contracting vectors is replaced by r, another set is obtained, listed below for future reference as well. The first set reads

$$\lambda_{rs}\lambda^s = \sigma_1\mu_r + k_1\nu_r \, , \quad \lambda_{rs}\mu^s = \sigma_2\mu_r + t_1\nu_r \, , \quad \lambda_{rs}\nu^s = \varepsilon_3\mu_r - \gamma_1\nu_r \, ; \tag{18a}$$

$$\mu_{rs}\lambda^s = -\sigma_1\lambda_r + t_1\nu_r \, , \quad \mu_{rs}\mu^s = -\sigma_2\lambda_r + k_2\nu_r \, , \quad \mu_{rs}\nu^s = -\varepsilon_3\lambda_r - \gamma_2\nu_r \, ; \tag{18b}$$

$$\nu_{rs}\lambda^{s} = -k_1\lambda_r - t_1\mu_r \ , \quad \nu_{rs}\mu^{s} = -t_1\lambda_r - k_2\mu_r \ , \quad \nu_{rs}\nu^{s} = \gamma_1\lambda_r + \gamma_2\mu_r \ . \tag{18c}$$

These equations could also be written with the index r raised.

Since it holds true (see e.g. Hotine's equation 3.19) that

$$\lambda_{rs}\lambda^{r} = \mu_{rs}\mu^{r} = \nu_{rs}\nu^{r} = 0 \ ,$$

the second set is limited to the following equations:

$$\lambda_{rs}\mu^{r} = -\mu_{rs}\lambda^{r} = \sigma_1\lambda_s + \sigma_2\mu_s + \varepsilon_3\nu_s \ , \tag{19a}$$

$$\lambda_{rs}\nu^{r} = -\nu_{rs}\lambda^{r} = k_1\lambda_s + t_1\mu_s - \gamma_1\nu_s \ , \tag{19b}$$

$$\mu_{rs}\nu^{r} = -\nu_{rs}\mu^{r} = t_1\lambda_s + k_2\mu_s - \gamma_2\nu_s \ . \tag{19c}$$

Contractions of (18a-c) with λ^{r}, μ^{r}, or ν^{r} recover all eight curvatures in (13a)-(17). The same can be said about (19a-c) in conjunction with λ^{s}, μ^{s}, or ν^{s}.

In analogy to the formulas (13a)-(19c) making use of space vectors, we present some of their counterparts in terms of surface tensors. In particular, of the eight curvatures listed in (13a)-(17), all except γ_1, γ_2, and ε_3 can be written as contractions of surface tensors. Such formulas are not necessary to provide new relationships in the current development, but can serve to verify the consistency and correctness of some results. The equation numbers of these and subsequent formulas will correspond to their numbers in the space context, except that they will be attributed a prime. In consulting equations (4.07)-(4.10) in [H], we write

$$\lambda_{\alpha\beta}\mu^{\alpha}\lambda^{\beta} = -\mu_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = \sigma_1 \ , \tag{13a'}$$

$$\lambda_{\alpha\beta}\mu^{\alpha}\mu^{\beta} = -\mu_{\alpha\beta}\lambda^{\alpha}\mu^{\beta} = \sigma_2 \ . \tag{13b'}$$

These relations are derived separately in Appendix C, equations (C.38,40). The formula for k_1 follows from Hotine's equation (7.03), while that for k_2 can be derived by similar means:

$$b_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta} = k_1 \ , \tag{14a'}$$

$$b_{\alpha\beta}\mu^{\alpha}\mu^{\beta} = k_2 \ ; \tag{14b'}$$

both are developed separately in Appendix C, equations (C.37,39).

In the same vein, the relations for t_1 are obtained from (7.08) in [H] as

$$b_{\alpha\beta}\lambda^{\alpha}\mu^{\beta} = b_{\alpha\beta}\mu^{\alpha}\lambda^{\beta} = t_1 \, , \tag{15'}$$

derived separately in Appendix C, equation (C.41). Hotine's equations (4.07), (4.10), (4.08), and (4.09) themselves yield

$$\lambda_{\alpha\beta}\lambda^{\beta} = \sigma_1\mu_{\alpha} \, , \qquad \lambda_{\alpha\beta}\mu^{\beta} = \sigma_2\mu_{\alpha} \; ; \tag{18a'}$$

$$\mu_{\alpha\beta}\lambda^{\beta} = -\sigma_1\lambda_{\alpha} \, , \qquad \mu_{\alpha\beta}\mu^{\beta} = -\sigma_2\lambda_{\alpha} \; . \tag{18b'}$$

These equations could also be written with the index α raised. Finally, equations (4.11) in [H] lead to

$$\lambda_{\alpha\beta}\mu^{\alpha} = -\mu_{\alpha\beta}\lambda^{\alpha} = \sigma_1\lambda_{\beta} + \sigma_2\mu_{\beta} \, , \tag{19a'}$$

which, in turn, lead to (13a') and (13b') upon the contractions with λ^{β} and μ^{β}, respectively.

3.2 Commutators for a General Function

We first introduce the concept of leg derivatives (scalar invariants) in conjunction with F, an arbitrary differentiable scalar function of position in space. In following the convention used in [Z], at a given point (e.g., P) we denote them by

$$F_{/1} \equiv F_r\lambda^r \, , \qquad F_{/2} \equiv F_r\mu^r \, , \qquad F_{/3} \equiv F_r\nu^r \, , \tag{20}$$

where $F_r \equiv \partial F/\partial x^r$. Accordingly, we can write

$$F_m = F_{/1}\lambda_m + F_{/2}\mu_m + F_{/3}\nu_m \; . \tag{21}$$

If $d\lambda$, $d\mu$, and $d\nu$ denote the length elements along λ, μ, and ν, the quantities $F_{/1}$, $F_{/2}$, and $F_{/3}$ in (20) are seen to be $\partial F/\partial\lambda$, $\partial F/\partial\mu$, and $\partial F/\partial\nu$, respectively. These leg derivatives are synonymous with directional derivatives along the directions of the 3-leg. Equation (21) falls in a general category of tensors expressed via λ, μ, and ν, such as

$$v_r = a_1\lambda_r + a_2\mu_r + a_3\nu_r \, ,$$

$$w_{rs} = a_{11}\lambda_r\lambda_s + a_{12}\lambda_r\mu_s + \dots \, ,$$

etc., where a_i, a_{ij}, are referred to in [Z] as "leg coefficients". These coefficients change only under leg changes, not under coordinate transformation. As is seen from (21), leg derivatives form a special class of leg coefficients.

Any one of the leg derivatives themselves can represent another scalar function "F", and can be treated in analogy to (21):

$$(F_{/i})_n = F_{/i/1}\lambda_n + F_{/i/2}\mu_n + F_{/i/3}\nu_n ; \quad i = 1,2,3 , \tag{22}$$

where

$$F_{/i/1} \equiv (F_{/i})_s\lambda^s , \qquad F_{/i/2} \equiv (F_{/i})_s\mu^s , \qquad F_{/i/3} \equiv (F_{/i})_s\nu^s . \tag{23}$$

Thus, for example,

$$F_{/1/2} \equiv (F_{/1})_s\mu^s = \partial(F_{/1})/\partial\mu = \partial/\partial\mu(\partial F/\partial\lambda) .$$

If the last equality were written as $\partial^2 F/\partial\lambda\partial\mu$, it could be confused with ordinary partial derivatives and changed into $\partial^2 F/\partial\mu\partial\lambda$. This, however, would be incorrect because the leg derivatives are not permutable as the ordinary partial derivatives. Such pitfalls, exposed in [Z], entice us to employ consistently the unambiguous leg-derivative notation. We remark that in [Z], the leg derivatives are defined in the section "The Hotine 3-leg and Commutators", and are further elaborated in Appendix A [ibid.].

When differentiated covariantly, equation (21) yields

$$F_{mn} = (F_{/1})_n\lambda_m + (F_{/2})_n\mu_m + (F_{/3})_n\nu_m + F_{/1}\lambda_{mn} + F_{/2}\mu_{mn} + F_{/3}\nu_{mn} . \tag{24}$$

If we contract (24), for example, with $\lambda^m\lambda^n$, we have

$$F_{mn}\lambda^m\lambda^n = F_{/1/1} + F_{/2}\mu_{mn}\lambda^m\lambda^n + F_{/3}\nu_{mn}\lambda^m\lambda^n ,$$

where the contractions in the last two terms on the right-hand side have already been presented in (13a) and (14a). In contracting (24) in succession with $\lambda^m\lambda^n$, $\lambda^m\mu^n$, etc., and consulting (13a)-(17), we obtain

$$F_{mn}\lambda^m\lambda^n = F_{/1/1} - \sigma_1 F_{/2} - k_1 F_{/3} , \tag{25a}$$

$$F_{mn}\lambda^m\mu^n = F_{/1/2} - \sigma_2 F_{/2} - t_1 F_{/3} , \tag{25b}$$

$$F_{mn}\lambda^m\nu^n = F_{/1/3} - \varepsilon_3 F_{/2} + \gamma_1 F_{/3} ; \tag{25c}$$

$$F_{mn}\mu^m\lambda^n = F_{/2/1} + \sigma_1 F_{/1} - t_1 F_{/3} \ . \tag{25a'}$$

$$F_{mn}\mu^m\mu^n = F_{/2/2} + \sigma_2 F_{/1} - k_2 F_{/3} \ . \tag{25b'}$$

$$F_{mn}\mu^m\nu^n = F_{/2/3} + \varepsilon_3 F_{/1} + \gamma_2 F_{/3} \ ; \tag{25c'}$$

$$F_{mn}\nu^m\lambda^n = F_{/3/1} + k_1 F_{/1} + t_1 F_{/2} \ . \tag{25a''}$$

$$F_{mn}\nu^m\mu^n = F_{/3/2} + t_1 F_{/1} + k_2 F_{/2} \ . \tag{25b''}$$

$$F_{mn}\nu^m\nu^n = F_{/3/3} - \gamma_1 F_{/1} - \gamma_2 F_{/2} \ . \tag{25c''}$$

In terms of the leg coefficients, F_{rs} can be written following the pattern of k_{rs} seen below (17):

$$\begin{aligned} F_{rs} = {} & (F_{mn}\lambda^m\lambda^n)\lambda_r\lambda_s + (F_{mn}\lambda^m\mu^n)\lambda_r\mu_s + (F_{mn}\lambda^m\nu^n)\lambda_r\nu_s \\ & + (F_{mn}\mu^m\lambda^n)\mu_r\lambda_s + (F_{mn}\mu^m\mu^n)\mu_r\mu_s + (F_{mn}\mu^m\nu^n)\mu_r\nu_s \\ & + (F_{mn}\nu^m\lambda^n)\nu_r\lambda_s + (F_{mn}\nu^m\mu^n)\nu_r\mu_s + (F_{mn}\nu^m\nu^n)\nu_r\nu_s \ , \end{aligned} \tag{26}$$

where the leg coefficients, listed in (25a-c"), are shown in parentheses. As has been indicated in conjunction with (6a,b) and (7), we formulate the commutators via

$$F_{rs} - F_{sr} = 0 \ . \tag{27}$$

If we generate a new equation by interchanging r and s in (26), and subtract this equation from (26), we deduce that the condition (27) is equivalent to

$$F_{mn}\lambda^m\mu^n - F_{mn}\mu^m\lambda^n = 0 \ ,$$

$$F_{mn}\nu^m\lambda^n - F_{mn}\lambda^m\nu^n = 0 \ ,$$

$$F_{mn}\mu^m\nu^n - F_{mn}\nu^m\mu^n = 0 \ .$$

The substitution of (25a-c") into these formulas yields the desired commutators:

$$F_{/1/2} - F_{/2/1} - \sigma_1 F_{/1} - \sigma_2 F_{/2} = 0 \ , \tag{28a}$$

$$F_{/3/1} - F_{/1/3} + k_1 F_{/1} + (t_1 + \varepsilon_3) F_{/2} - \gamma_1 F_{/3} = 0 \ , \tag{28b}$$

$$F_{/2/3} - F_{/3/2} - (t_1 - \varepsilon_3) F_{/1} - k_2 F_{/2} + \gamma_2 F_{/3} = 0 \ . \tag{28c}$$

Equations (28a-c) are identical to the commutators listed in (2) of [Z].

As an added benefit of this demonstration, we can formulate ΔF, the Laplacian of F, with almost no additional effort. From the definition

$$\Delta F = g^{rs} F_{rs} .$$

and from the leg formulation of the associated metric tensor, namely

$$g^{rs} = \lambda^r \lambda^s + \mu^r \mu^s + \nu^r \nu^s ,$$

one has

$$\Delta F = F_{rs} \lambda^r \lambda^s + F_{rs} \mu^r \mu^s + F_{rs} \nu^r \nu^s . \qquad (29)$$

The same result follows upon applying g^{rs} directly (without the leg formulation) to (26). With (25a-c") equation (29) yields

$$\Delta F = F_{/1/1} + F_{/2/2} + F_{/3/3} - (\gamma_1 - \sigma_2) F_{/1} - (\gamma_2 + \sigma_1) F_{/2} - 2H F_{/3} , \qquad (30)$$

where

$$2H = k_1 + k_2 ,$$

and where H is known as the mean curvature, which need not be confused with the parameter H (generalized latitude).

4. LEG DERIVATIVES AND COMMUTATORS IN TERMS OF FIVE CURVATURE PARAMETERS

4.1 Leg Derivatives of Curvatures in General Terms

In this section we present formulas giving pairwise combinations (here sums or differences) of the leg derivatives of curvatures in a general space. As an example, we develop $k_{1/2}-t_{1/1}$ along two separate paths. The initial equations in the first path are (14a) and (15), where the first alternatives are utilized (containing λ_{rs}), namely

$$k_1 = \lambda_{rs}\nu^r\lambda^s \;, \qquad t_1 = \lambda_{rs}\nu^r\mu^s \;.$$

These equations yield

$$k_{1/2} \equiv (k_1)_t\mu^t = \lambda_{rst}\nu^r\lambda^s\mu^t + \lambda_{rs}\nu^r_t\lambda^s\mu^t + \lambda_{rs}\nu^r\lambda^s_t\mu^t \;,$$

$$t_{1/1} \equiv (t_1)_t\lambda^t = \lambda_{rst}\nu^r\mu^s\lambda^t + \lambda_{rs}\nu^r_t\mu^s\lambda^t + \lambda_{rs}\nu^r\mu^s_t\lambda^t \;,$$

where ν^r_t could also be written as $\nu^r_{.t}$, etc. After an exchange of the indices s and t in the first of the three terms comprising $t_{1/1}$, the required combination $k_{1/2}-t_{1/1}$ is seen to contain a contraction of the covariant R-tensor:

$$(\lambda_{rst}-\lambda_{rts})\nu^r\lambda^s\mu^t = (R^m_{.rst}\lambda_m)\nu^r\lambda^s\mu^t = R_{urst}\lambda^u\nu^r\lambda^s\mu^t \;,$$

which follows from (5.02) and (5.06) in [H]. As usual, this and subsequent tensor equations are considered at the point P.

Since contractions of the covariant R-tensor with four contravariant vectors of the general 3-leg occur in each of the required combinations, it is useful to simplify the notation. Upon defining the ranking of λ, μ, and ν as the first, second, and third, respectively, the adopted symbolism lists in parentheses the ranking of the vector attributed the first index of the covariant R-tensor (below listed as u), followed by the ranking of the vector attributed the second index (below listed as r), etc. This convention is exemplified by

$$R_{urst}\lambda^u\nu^r\lambda^s\mu^t \equiv R(1,3,1,2) \;,$$

which is the first term in the execution of $k_{1/2}-t_{1/1}$. In expressing the other terms, one draws on the first two formulas from (18a), on the first formula from

(18b), on the first two formulas from (18c), and on (19b). Accordingly, the result is

$$k_{1/2} - t_{1/1} = R(1,3,1,2) + (k_1 - k_2)\sigma_1 + 2t_1\sigma_2 .$$

The second path, used as a means of verification, proceeds via contractions of surface tensors. In the present case, we have (14a') and the first alternative of (15') as the initial equations:

$$k_1 = b_{\alpha\beta}\lambda^\alpha\lambda^\beta , \qquad t_1 = b_{\alpha\beta}\lambda^\alpha\mu^\beta .$$

In analogy to the above procedure, we form

$$k_{1/2} \equiv (k_1)_t\mu^t = (k_1)_\gamma\mu^\gamma = b_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\mu^\gamma + b_{\alpha\beta}\lambda^\alpha_\gamma\lambda^\beta\mu^\gamma + b_{\alpha\beta}\lambda^\alpha\lambda^\beta_\gamma\mu^\gamma ,$$

$$t_{1/1} \equiv (t_1)_t\lambda^t = (t_1)_\gamma\lambda^\gamma = b_{\alpha\beta\gamma}\lambda^\alpha\mu^\beta\lambda^\gamma + b_{\alpha\beta}\lambda^\alpha_\gamma\mu^\beta\lambda^\gamma + b_{\alpha\beta}\lambda^\alpha\mu^\beta_\gamma\lambda^\gamma .$$

After an exchange of the indices β and γ in the first of the three terms comprising $t_{1/1}$, the required combination will involve Hotine's formula (6.22):

$$b_{\alpha\beta\gamma} - b_{\alpha\gamma\beta} = -R_{urst}\nu^u x^r_\alpha x^s_\beta x^t_\gamma .$$

Upon employing identities corresponding to (6.07) in [H], such as

$$x^r_\alpha h^\alpha = h^r ,$$

where h is a surface vector, the first term in the result for $k_{1/2}-t_{1/1}$ is seen to be

$$-R_{urst}\nu^u\lambda^r\lambda^s\mu^t = R_{urst}\lambda^u\nu^r\lambda^s\mu^t = R(1,3,1,2),$$

the same as that found in the previous paragraph. In expressing the other terms, we draw on (18a') and the first formula from (18b'), and on (14a')-(15'). The final result for $k_{1/2}-t_{1/1}$ is again identical to the one presented in the preceding paragraph.

In terms of space vectors, a required combination of the leg derivatives of curvatures is derived upon first performing the *covariant differentiation* of the two pertinent curvatures from suitable alternatives in (13a)-(17), contracting either outcome with the pertinent contravariant leg vector, interchanging the second and the third indices in one of the two terms containing second covariant derivatives, and subtracting one equation from the other. Subsequently, the difference between the second covariant derivatives of a vector in the resulting equation is expressed via the covariant R-tensor; the latter is always contracted with a permutation of four contravariant vectors of the general 3-leg. Finally, all of the remaining terms are obtained with the aid of (18a-c) and (19a-c). With regard to the (verification) procedure using surface tensors, its description is similar except that it involves (13a')-(15') instead of (13a)-(17), (18a',b') instead of (18a-c), (19a') instead of (19a-c), and, additionally, Hotine's formula (6.22).

The required combinations derived in this manner are listed as follows:

$$k_{1/2} - t_{1/1} = R(1,3,1,2) + (k_1-k_2)\sigma_1 + 2t_1\sigma_2 \, , \tag{31a}$$

$$k_{1/3} + \gamma_{1/1} = R(1,3,1,3) + k_1^2 + t_1^2 + \gamma_1^2 + 2t_1\varepsilon_3 + \gamma_2\sigma_1 \, , \tag{31b}$$

$$t_{1/3} + \gamma_{1/2} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 - (k_1-k_2)\varepsilon_3 + \gamma_2\sigma_2 \, , \tag{31c}$$

$$t_{1/2} - k_{2/1} = R(2,3,1,2) - (k_1-k_2)\sigma_2 + 2t_1\sigma_1 \, , \tag{31d}$$

$$t_{1/3} + \gamma_{2/1} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 - (k_1-k_2)\varepsilon_3 - \gamma_1\sigma_1 \, , \tag{31e}$$

$$k_{2/3} + \gamma_{2/2} = R(2,3,2,3) + k_2^2 + t_1^2 + \gamma_2^2 - 2t_1\varepsilon_3 - \gamma_1\sigma_2 \, , \tag{31f}$$

$$\sigma_{1/2} - \sigma_{2/1} = R(1,2,1,2) + k_1k_2 - t_1^2 + \sigma_1^2 + \sigma_2^2 \, , \tag{31g}$$

$$\sigma_{1/3} - \varepsilon_{3/1} = R(1,3,1,2) - k_1\gamma_2 + t_1\gamma_1 + k_1\sigma_1 + t_1\sigma_2 - \gamma_1\varepsilon_3 + \varepsilon_3\sigma_2 \, , \tag{31h}$$

$$\sigma_{2/3} - \varepsilon_{3/2} = R(2,3,1,2) + k_2\gamma_1 - t_1\gamma_2 + t_1\sigma_1 + k_2\sigma_2 - \gamma_2\varepsilon_3 - \varepsilon_3\sigma_1 \, . \tag{31i}$$

In the approach using space vectors, the initial equations have been: first alternatives in (14a) and (15) for (31a); first alternatives in (14a) and (16a) for (31b); first alternatives in (15) and (16a) for (31c); first alternative in (14b) and third alternative in (15) for (31d); third alternative in (15) and first alternative in (16b) for (31e); first alternatives in (14b) and (16b) for (31f); first alternatives in (13a,b) for (31g); first alternatives in (13a) and (17) for (31h); and first alternatives in (13b) and (17) for (31i). In the (verification) approach using surface tensors, the initial equations have been: (14a') and first alternative in (15') for (31a); (14b') and second alternative in (15') for (31d); and first alternatives in (13a',b') for (31g). A detailed derivation of (31a-i), including the verification approach, is presented in Appendix D.

A few remarks are in order with regard to (31a-i). Upon differencing (31a) and (31h), (31c) and (31e), and (31d) and (31i), the R-terms are eliminated and the resulting identities, valid in any space, are thereby simplified. As a matter of verification, we notice that the identity produced upon differencing (31c) and (31e), namely

$$\gamma_{1/2} - \gamma_{2/1} = \gamma_1\sigma_1 + \gamma_2\sigma_2 , \tag{32}$$

could be derived by other means. In particular, if we use the definitions

$$\gamma_1 = (1/n)n_t\lambda^t \equiv (1/n)n_{/1} , \qquad \gamma_2 = (1/n)n_t\mu^t \equiv (1/n)n_{/2} , \tag{32'}$$

express the appropriate leg derivatives, difference the resulting relations, and make use of the second formula in (18a) and the first formula in (18b), we recover (32). In the process, the relation $n_{rs}=n_{sr}$ has been utilized. This is justified since N and thus also n are proper invariants by definition and, accordingly, have symmetric second-order covariant derivatives in any space.

Next, we observe that if the pertinent R-terms are zero, (31a) and (31d) are essentially Hotine's equations (8.23), referred to as "another form" of the Mainardi-Codazzi equations. Clearly, if the space is flat and thus the covariant R-tensor is identically zero, these Mainardi-Codazzi equations are confirmed. However, it is now seen that they hold true also in a class of curved spaces, where R(1,3,1,2)=0 and R(2,3,1,2)=0. We comment that no generality has been lost by considering a special pair of orthonormal surface vectors (λ and μ), since, if these two R-terms are zero for one such pair, they are zero for all pairs. Indeed, if we express a general pair $\ell^r=\lambda^r\cos\theta+\mu^r\sin\theta$, $j^r=-\lambda^r\sin\theta+\mu^r\cos\theta$, and stipulate that $R_{urst}\ell^u\nu^r\ell^s j^t=0$ and $R_{urst}j^u\nu^r\ell^s j^t=0$, we notice that, for any θ, either stipulation entails no constraints other than the above R(1,3,1,2)=0 and R(2,3,1,2)=0. We also comment that the combinations of the leg derivatives of curvatures are expressed on the right-hand sides of (31a-i) in terms of all eight curvatures. We will be able to express them in terms of the five curvature parameters only after linking σ_1, σ_2, and ε_3 to these curvature parameters via G and H.

Finally, we note that if all the contractions of the covariant R-tensor in (31a-i) were zero, these equations would correspond respectively to the quantities (ω_I), (ω_{II}), (ω_{III}), (ϕ_I), (ϕ_{II}), (ϕ_{III}), (ω^*_I), (ω^*_{II}), and (ω^*_{III})

identified in the last nine equations listed in the section "Hotine's Problem Re-examined" of [Z]. However, in the case of our equations (31d-f), the correspondence with (ϕ_I), (ϕ_{II}), and (ϕ_{III}) of [Z] is achieved only after using the identities (10) [ibid.], which "hold only upon introduction of (ω,ϕ) as coordinates on the N-surface" [ibid.]. Thus, even if the pertinent contractions of the covariant R-tensor were zero, our equations (31d-f) would still be valid in a more general environment than their counterparts in [Z]. Since Zund confines his analysis to the flat space from the outset, implying the vanishing of the covariant R-tensor, and since he purposefully introduces the coordinates ω and ϕ at an early stage, his development leads to the nine quantities (ω_I)-(ω^*_{III}) as listed in [Z].

The distinction between (ω_I)-(ω^*_{III}) and the above equation set (31a-i) underlies the distinction between the development in [Z], confined to the flat space and concerned with "Hotine's assertion", and the development herein, which proceeds in a space of general curvature characteristics and delays the introduction of the first two space coordinates (generalized to G and H) for a later stage. We remark that if our analysis were confined to the flat space from the outset, and if this (global) flatness were to be guaranteed, then we would be compelled not only to have the covariant R-tensor at P equal to zero, but *all* of its spatial derivatives at P equal to zero as well. Equivalently, we would require that the covariant R-tensor be zero at *every point of the space* (see the latter part of Section A.2 in Appendix A).

4.2 Leg Derivatives of G and H in Terms of Five Curvature Parameters

In order to link the parameters G and H to the curvatures, we return to (11a-c) and contract these equations respectively with λ^r, μ^r, and ν^r. Of the resulting system, three equations are identically zero, three are in general different from zero, and the remaining three are repetitious. The three non-repetitious equations form the system

$$\begin{bmatrix} \lambda_{rs}\mu^r \\ \lambda_{rs}\nu^r \\ \mu_{rs}\nu^r \end{bmatrix} = \begin{bmatrix} T_{12} & T'_{12} \\ T_{13} & T'_{13} \\ T_{23} & T'_{23} \end{bmatrix} \begin{bmatrix} G_s \\ H_s \end{bmatrix} \equiv T \begin{bmatrix} G_s \\ H_s \end{bmatrix} . \qquad (33)$$

Each of the three entries on the left-hand side represents a row vector of three elements, and the left-hand side is accordingly a matrix of dimensions (3×3). The entity inside the first pair of brackets on the right-hand side is a matrix of dimensions (3×2) as indicated, denoted T. And the last entity in brackets on the right-hand side is a matrix of dimensions (2×3), where G_s and H_s represent row vectors of three elements each.

In accordance with the uniqueness of G and H (in conjunction with a given 3-leg), the partial derivatives G_s and H_s are also assumed unique at P . Thus, the matrix T has the full column rank 2. In consulting (19a-c), we observe that the first row on the left-hand side of (33) is formed through the curvatures σ_1, σ_2, and ε_3, whereas the second and the third rows are formed through the five curvature parameters k_1, k_2, t_1, γ_1, and γ_2. Accordingly, should G_s and H_s be expressible via these curvature parameters, the submatrix of T comprising the second and the third rows must be nonsingular. We henceforth make this assumption and discard the cases where it would not hold true.

We further discard the case where T_{12} and T'_{12} are both zero, which would imply that all of σ_1, σ_2, and ε_3 are zero. This indicates that at least one other nonsingular submatrix of T must exist. Suppose that the submatrix formed by the first and the second rows of T is singular, so that the first row is a linear combination of the second. Then the submatrix formed by the first and the third rows must be nonsingular because otherwise the third row would be a linear combination of the first and thereby also of the second, contrary to the present situation. We denote the determinant of the submatrix formed by the second and the third rows of T by the letter D, and refer to the solution produced by the corresponding subsystem as the "main solution". Of the other two submatrices of T, we assume that the assuredly nonsingular one is formed by the first and the third rows, denote its determinant by D', and refer to the solution produced by the corresponding subsystem as the "alternate solution".

Finally, we denote the determinant of the submatrix formed by the first and the second rows of T by D". We make no assumption as to the singularity or nonsingularity of this submatrix, and do not resolve the corresponding subsystem. (If it turned out, in some cases, that $D'=0$ but $D''\neq 0$, we could interchange the second and the third equations in the above system 33 and proceed in analogy to the current development.) In summary, we have

$$D = T_{13}T'_{23} - T'_{13}T_{23} \neq 0 \,, \tag{34a}$$

$$D' = T_{12}T'_{23} - T'_{12}T_{23} \neq 0 \,, \tag{34b}$$

$$D'' = T_{12}T'_{13} - T'_{12}T_{13} \,; \tag{34c}$$

the subsystem associated with D leads to the main solution, while the subsystem associated with D' leads to the alternate solution. We will formulate the commutators and resolve as many questions as possible using the main solution.

If we solve for G_s and H_s from the main subsystem consisting of the second and the third equations of the system (33), and contract them in turn by λ^s, μ^s, and ν^s, we obtain the following leg derivatives of the parameters G and H:

$$G_{/1} = (k_1T'_{23}-t_1T'_{13})/D \,, \qquad G_{/2} = (t_1T'_{23}-k_2T'_{13})/D \,,$$
$$G_{/3} = (-\gamma_1T'_{23}+\gamma_2T'_{13})/D \,. \tag{35}$$

$$H_{/1} = (-k_1T_{23}+t_1T_{13})/D \,, \qquad H_{/2} = (-t_1T_{23}+k_2T_{13})/D \,,$$
$$H_{/3} = (\gamma_1T_{23}-\gamma_2T_{13})/D \,. \tag{36}$$

As is seen above, the formulas giving $H_{/i}$, i=1,2,3, are obtained from those giving $G_{/i}$ upon the following replacements:

$$T'_{23} \to -T_{23} \,, \qquad T'_{13} \to -T_{13} \,. \tag{36'}$$

This simple relationship will be instrumental in providing shortcuts in the derivations. If we insert the solution for G_s and H_s into the first (dependent) equation of the system (33), we obtain identities for σ_1, σ_2, and ε_3 in terms of the five curvature parameters:

$$\sigma_1 = (k_1D'-t_1D'')/D \,, \quad \sigma_2 = (t_1D'-k_2D'')/D \,, \quad \varepsilon_3 = (-\gamma_1D'+\gamma_2D'')/D \,. \tag{37}$$

These identities will enable us to express, for example, the right-hand sides of (31a-i) in terms of the five curvature parameters.

In developing the double leg derivatives from (35), and forming $G_{/m/n} - G_{/n/m}$, we first utilize the general relation

$$(fM)_{/n} = f_{/n}M + fM_{/n} \,,$$

and subsequently use a more specialized relation

$$(T'_{ij}/D)_{/n} = [\partial(T'_{ij}/D)/\partial G]G_{/n} + [\partial(T'_{ij}/D)/\partial H]H_{/n} \,.$$

where each T'_{ij} may be replaced by T_{ij}. To streamline the derivations, we denote

$$k_1 \equiv a_1 , \quad t_1 \equiv a_2 , \quad -\gamma_1 \equiv a_3 , \quad t_1 \equiv b_1 , \quad k_2 \equiv b_2 , \quad -\gamma_2 \equiv b_3 , \qquad (38)$$

and write (35) in a compact form as

$$G_{/m} = (a_m T'_{23} - b_m T'_{13})/D .$$

This form serves with advantage in expressing $H_{/m}$, which follows readily from $G_{/m}$ in accordance with (36'). The benefit to the derivations emerges, for example, in conjunction with the intermediate results

$$a_m G_{/n} - a_n G_{/m} = -(a_m b_n - a_n b_m) T'_{13}/D ,$$

$$b_m G_{/n} - b_n G_{/m} = -(a_m b_n - a_n b_m) T'_{23}/D ,$$

which, due to (36'), can be subsequently applied to the leg derivatives of H.

From the formulas, notation, and intermediate results outlined in the preceding paragraph, straightforward algebra leads to the general expression

$$\begin{aligned} G_{/m/n} - G_{/n/m} = \{ & [a_{m/n} - a_{n/m} + (a_m b_n - a_n b_m) K_{13}/D] T'_{23} \\ & - [b_{m/n} - b_{n/m} + (a_m b_n - a_n b_m) K_{23}/D] T'_{13} \}/D, \end{aligned} \qquad (39)$$

where

$$K_{13} = \partial T'_{13}/\partial G - \partial T_{13}/\partial H , \qquad K_{23} = \partial T'_{23}/\partial G - \partial T_{23}/\partial H . \qquad (40a,b)$$

We reinstate the curvature parameters from (38), and specialize (39) for m=1, n=2, then for m=3, n=1, and finally for m=2, n=3. This yields the differences of the double leg derivatives of G in terms of the five curvature parameters:

$$\begin{aligned} G_{/1/2} - G_{/2/1} = \{ & [k_{1/2} - t_{1/1} + (k_1 k_2 - t_1^2) K_{13}/D] T'_{23} \\ & - [t_{1/2} - k_{2/1} + (k_1 k_2 - t_1^2) K_{23}/D] T'_{13} \}/D , \end{aligned} \qquad (41a)$$

$$\begin{aligned} G_{/3/1} - G_{/1/3} = \{ & -[k_{1/3} + \gamma_{1/1} + (t_1 \gamma_1 - k_1 \gamma_2) K_{13}/D] T'_{23} \\ & + [t_{1/3} + \gamma_{2/1} + (t_1 \gamma_1 - k_1 \gamma_2) K_{23}/D] T'_{13} \}/D , \end{aligned} \qquad (41b)$$

$$\begin{aligned} G_{/2/3} - G_{/3/2} = \{ & [t_{1/3} + \gamma_{1/2} + (k_2 \gamma_1 - t_1 \gamma_2) K_{13}/D] T'_{23} \\ & - [k_{2/3} + \gamma_{2/2} + (k_2 \gamma_1 - t_1 \gamma_2) K_{23}/D] T'_{13} \}/D . \end{aligned} \qquad (41c)$$

These formulas apply in a general space, hence we do not replace $k_1 k_2 - t_1^2$ by K, the Gaussian curvature, which would be valid only in the flat space. The expressions for $H_{/m/n}-H_{/n/m}$ can be transcribed from (39) and, in particular, from (41a-c) upon recalling (36'); there is no need to carry out this direct transcription explicitly. All such differences of the double leg derivatives can be expressed through the five curvature parameters if one substitutes the appropriate combinations of the leg derivatives of curvatures from the relations developed below, in particular, from (42a-f).

4.3 Leg Derivatives of Curvatures in Terms of Five Curvature Parameters

In using (37) in straightforward substitutions, we reformulate (31a-i) in terms of the five curvature parameters. This, together with (41a-c) and similar relations for H, will enable us to present also the commutators in terms of the five curvature parameters. The reformulated results are

$$k_{1/2} - t_{1/1} = R(1,3,1,2) - [(k_1k_2-2t_1^2-k_1^2)D' + 2Ht_1D'']/D \, , \tag{42a}$$

$$k_{1/3} + \gamma_{1/1} = R(1,3,1,3) + k_1^2 + t_1^2 + \gamma_1^2 + [(k_1\gamma_2-2t_1\gamma_1)D' + t_1\gamma_2D'']/D \, , \tag{42b}$$

$$t_{1/3} + \gamma_{1/2} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 + [(k_1\gamma_1-k_2\gamma_1+t_1\gamma_2)D' - k_1\gamma_2D'']/D \, , \tag{42c}$$

$$t_{1/2} - k_{2/1} = R(2,3,1,2) + [2Ht_1D' + (k_1k_2-2t_1^2-k_2^2)D'']/D \, , \tag{42d}$$

$$t_{1/3} + \gamma_{2/1} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 - [k_2\gamma_1D' + (k_1\gamma_2-k_2\gamma_2-t_1\gamma_1)D'']/D \, , \tag{42e}$$

$$k_{2/3} + \gamma_{2/2} = R(2,3,2,3) + k_2^2 + t_1^2 + \gamma_2^2 + [t_1\gamma_1D' + (k_2\gamma_1-2t_1\gamma_2)D'']/D \, , \tag{42f}$$

$$\sigma_{1/2} - \sigma_{2/1} = R(1,2,1,2) + k_1k_2 - t_1^2 + [(k_1^2+t_1^2)D'^2 - 4Ht_1D'D'' + (k_2^2+t_1^2)D''^2]/D^2 \, , \tag{42g}$$

$$\sigma_{1/3} - \varepsilon_{3/1} = R(1,3,1,2) - k_1\gamma_2 + t_1\gamma_1 + [(k_1^2+t_1^2+\gamma_1^2)D' - (2Ht_1+\gamma_1\gamma_2)D'']/D + [-t_1\gamma_1D'^2 + (k_2\gamma_1+t_1\gamma_2)D'D'' - k_2\gamma_2D''^2]/D^2 \, . \tag{42h}$$

$$\sigma_{2/3} - \varepsilon_{3/2} = R(2,3,1,2) + k_2\gamma_1 - t_1\gamma_2 + [(2Ht_1+\gamma_1\gamma_2)D' - (k_2^2+t_1^2+\gamma_2^2)D'']/D + [k_1\gamma_1 D'^2 - (k_1\gamma_2+t_1\gamma_1)D'D'' + t_1\gamma_2 D''^2]/D^2 \ . \tag{42i}$$

4.4 Commutators in Terms of Five Curvature Parameters

We are now in a position to formulate the desired commutators; we begin by transcribing (28a-c) for G:

$$G_{/1/2} - G_{/2/1} - \sigma_1 G_{/1} - \sigma_2 G_{/2} = 0 \ , \tag{43a}$$

$$G_{/3/1} - G_{/1/3} + k_1 G_{/1} + (t_1+\varepsilon_3)G_{/2} - \gamma_1 G_{/3} = 0 \ , \tag{43b}$$

$$G_{/2/3} - G_{/3/2} - (t_1-\varepsilon_3)G_{/1} - k_2 G_{/2} + \gamma_2 G_{/3} = 0 \ . \tag{43c}$$

We next substitute (41a-c) for the differences of the double leg derivatives of G, into which we first substitute (42a-f) giving combinations of the leg derivatives of curvatures; substitute (35) for the (single) leg derivatives of G; and substitute (37) for σ_1, σ_2, and ε_3. It is unnecessary to carry out these tasks for H, due to the fact that both the leg derivatives (in equation 36) and the differences of the double leg derivatives (past equations 41a-c) follow from their counterparts for G upon the replacements (36'). The same then holds true for the corresponding commutators. Since all of the above substitutions are expressed in terms of the five curvature parameters, so are the resulting commutators for G and H. In performing the operations as just indicated, for the G-commutators we deduce

$$[R(1,3,1,2) + (k_1k_2-t_1^2)(K_{13}-D')/D]T'_{23} - [R(2,3,1,2) + (k_1k_2-t_1^2)(K_{23}+D'')/D]T'_{13} = 0 \ , \tag{44a}$$

$$-[R(1,3,1,3) + (t_1\gamma_1-k_1\gamma_2)(K_{13}-D')/D]T'_{23} + [R(2,3,1,3) + (t_1\gamma_1-k_1\gamma_2)(K_{23}+D'')/D]T'_{13} = 0 \ . \tag{44b}$$

$$[R(2,3,1,3) + (k_2\gamma_1-t_1\gamma_2)(K_{13}-D')/D]T'_{23} - [R(2,3,2,3) + (k_2\gamma_1-t_1\gamma_2)(K_{23}+D'')/D]T'_{13} = 0 \ . \tag{44c}$$

The three H-commutators are obtained from (44a-c) upon the replacements (36').

In abbreviated notation, the three G-commutators read

$$U_1 T'_{23} - V_1 T'_{13} = 0 , \quad -U_2 T'_{23} + V_2 T'_{13} = 0 , \quad U_3 T'_{23} - V_3 T'_{13} = 0 . \tag{45a}$$

where the terms U_i and V_i, i=1,2,3, are easily identified from (44a-c). The H-commutators are similarly written as

$$-U_1 T_{23} + V_1 T_{13} = 0 , \quad U_2 T_{23} - V_2 T_{13} = 0 , \quad -U_3 T_{23} + V_3 T_{13} = 0 . \tag{45b}$$

Upon assuming that $T'_{23} \neq 0$, (45a,b) yield

$$-V_1 D = 0 , \quad -V_2 D = 0 , \quad -V_3 D = 0 .$$

If this assumption is invalid, but $T'_{13} \neq 0$ holds true, it follows that

$$-U_1 D = 0 , \quad -U_2 D = 0 , \quad -U_3 D = 0 .$$

Since $D \neq 0$ (see equation 34a), at least one of the two assumptions must be true; in either case we obtain

$$U_1 = U_2 = U_3 = 0 , \qquad V_1 = V_2 = V_3 = 0 . \tag{46a,b}$$

Finally, we present the results (46a,b) explicitly as

$$R(1,3,1,2) + (k_1 k_2 - t_1^2)(K_{13} - D')/D = 0 , \tag{47a}$$

$$R(1,3,1,3) + (t_1 \gamma_1 - k_1 \gamma_2)(K_{13} - D')/D = 0 , \tag{47b}$$

$$R(2,3,1,3) + (k_2 \gamma_1 - t_1 \gamma_2)(K_{13} - D')/D = 0 ; \tag{47c}$$

$$R(2,3,1,2) + (k_1 k_2 - t_1^2)(K_{23} + D'')/D = 0 , \tag{47d}$$

$$R(2,3,1,3) + (t_1 \gamma_1 - k_1 \gamma_2)(K_{23} + D'')/D = 0 , \tag{47e}$$

$$R(2,3,2,3) + (k_2 \gamma_1 - t_1 \gamma_2)(K_{23} + D'')/D = 0 . \tag{47f}$$

Equations (47a-f) are equivalent to six commutators for G and H in terms of the five curvature parameters and of the contracted covariant R-tensor. We express the latter conveniently in a temporary coordinate system at P, which we define as locally Cartesian, but with the coordinate axes directed along the triad λ, μ, ν instead of A, B, C. In this system, the contractions R(1,3,1,2), etc., are nothing else but R_{1312}, etc., that is, actual components of the covariant R-tensor. Since the latter has only six independent components, six equations can be used to resolve it. Unfortunately, only five independent components of this tensor are represented by (47a-f), due to both (47c) and (47e) containing R(2,3,1,3). In order to gather one additional independent relation in view of the covariant R-tensor, we are compelled to resort to the alternate solution.

4.5 Alternate Solution

If we solve for G_s and H_s from the alternate subsystem consisting of the first and the third equations of (33), and contract them in turn by λ^s, μ^s, and ν^s, we obtain two sets of leg derivatives, the first of which reads

$$G_{/1} = (\sigma_1 T'_{23} - t_1 T'_{12})/D', \qquad G_{/2} = (\sigma_2 T'_{23} - k_2 T'_{12})/D',$$
$$G_{/3} = (\varepsilon_3 T'_{23} + \gamma_2 T'_{12})/D'. \qquad (48)$$

The three leg derivatives of H follow from (48) upon the replacements

$$T'_{23} \rightarrow -T_{23}, \qquad T'_{12} \rightarrow -T_{12}. \qquad (48')$$

In analogy to (38), we denote

$$\sigma_1 \equiv a_1, \quad \sigma_2 \equiv a_2, \quad \varepsilon_3 \equiv a_3, \quad t_1 \equiv b_1, \quad k_2 \equiv b_2, \quad -\gamma_2 \equiv b_3.$$

In pursuing a path paralleling the development that followed (37), we arrive at a relation for $G_{/m/n} - G_{/n/m}$ similar to (39), where, however, D is replaced by D' (in three instances), T'_{13} is replaced by T'_{12}, and K_{13} is replaced by K_{12}; the latter is defined as

$$K_{12} = \partial T'_{12}/\partial G - \partial T_{12}/\partial H. \qquad (49)$$

The specializations for n, m yield the formulas for $G_{/1/2} - G_{/2/1}$, $G_{/3/1} - G_{/1/3}$, and $G_{/2/3} - G_{/3/2}$. However, unlike (41a-c), these formulas contain all eight curvatures, i.e., contain σ_1, σ_2, and ε_3 in addition to the five curvature parameters. The same can be said about the formulas for $H_{/m/n} - H_{/n/m}$, which are obtained from their counterparts for G upon the replacements (48').

The double leg derivatives just mentioned can be rid of the curvatures σ_1, σ_2, and ε_3 upon applying the identities (37) transformed into the following useful form:

$$(k_2 \sigma_1 - t_1 \sigma_2) D = (k_1 k_2 - t_1^2) D'. \qquad (50a)$$

$$(t_1\varepsilon_3+\gamma_2\sigma_1)D = (k_1\gamma_2-t_1\gamma_1)D' \ , \tag{50b}$$

$$(k_2\varepsilon_3+\gamma_2\sigma_2)D = -(k_2\gamma_1-t_1\gamma_2)D' \ . \tag{50c}$$

These equations lead to the results paralleling (41a-c), namely

$$G_{/1/2} - G_{/2/1} = \{[\sigma_{1/2} - \sigma_{2/1} + (k_1k_2-t_1^2)K_{12}/D]T'_{23} - [t_{1/2} - k_{2/1} + (k_1k_2-t_1^2)K_{23}/D]T'_{12}\}/D' \ , \tag{51a}$$

$$G_{/3/1} - G_{/1/3} = \{-[\sigma_{1/3} - \varepsilon_{3/1} + (t_1\gamma_1-k_1\gamma_2)K_{12}/D]T'_{23} + [t_{1/3} + \gamma_{2/1} + (t_1\gamma_1-k_1\gamma_2)K_{23}/D]T'_{12}\}/D' \ , \tag{51b}$$

$$G_{/2/3} - G_{/3/2} = \{[\sigma_{2/3} - \varepsilon_{3/2} + (k_2\gamma_1-t_1\gamma_2)K_{12}/D]T'_{23} - [k_{2/3} + \gamma_{2/2} + (k_2\gamma_1-t_1\gamma_2)K_{23}/D]T'_{12}\}/D' \ . \tag{51c}$$

The formulas for H are obtained upon the replacements (48'). All of these differences of the double leg derivatives can be expressed through the five curvature parameters if one substitutes the appropriate relations in terms of the leg derivatives of curvatures from (42d-i).

To obtain the G-commutators, we recall equations (43a-c) and make the following changes: substitute (51a-c) for the differences of the double leg derivatives of G, into which we first substitute (42d-i) giving combinations of the leg derivatives of curvatures; substitute (48) for the leg derivatives of G; and substitute (37) for σ_1, σ_2, and ε_3. This yields

$$[R(1,2,1,2) + (k_1k_2-t_1^2)(K_{12}+D)/D]T'_{23} - [R(2,3,1,2) + (k_1k_2-t_1^2)(K_{23}+D'')/D]T'_{12} = 0 \ . \tag{52a}$$

$$-[R(1,3,1,2) + (t_1\gamma_1-k_1\gamma_2)(K_{12}+D)/D]T'_{23} + [R(2,3,1,3) + (t_1\gamma_1-k_1\gamma_2)(K_{23}+D'')/D]T'_{12} = 0 \ , \tag{52b}$$

$$[R(2,3,1,2) + (k_2\gamma_1-t_1\gamma_2)(K_{12}+D)/D]T'_{23} - [R(2,3,2,3) + (k_2\gamma_1-t_1\gamma_2)(K_{23}+D'')/D]T'_{12} = 0 \ . \tag{52c}$$

The three H-commutators are obtained from (52a-c) upon the replacements (48'). The derivation of (52a-c) could have been slightly shorter if σ_1, σ_2, and ε_3 had been replaced at a later stage. Instead of using (51a-c) in the substitution described above, we could have used the formulas mentioned (but not listed) below (49); into these we could have substituted (31d-i) rather than (42d-i); and to the outcome we could have applied (50a-c), thereby by-passing (37).

Having found the six commutators, we follow the procedure that has led from (45a) to (46b), in which $D \neq 0$ is now replaced by $D' \neq 0$. Obtained in this manner, the first three of six equations paralleling (47a-f) are presented as

$$R(1,2,1,2) + (k_1 k_2 - t_1^2)(K_{12}+D)/D = 0 \ , \tag{53a}$$

$$R(1,3,1,2) + (t_1 \gamma_1 - k_1 \gamma_2)(K_{12}+D)/D = 0 \ . \tag{53b}$$

$$R(2,3,1,2) + (k_2 \gamma_1 - t_1 \gamma_2)(K_{12}+D)/D = 0 \ . \tag{53c}$$

The remaining three equations turn out to be (47d-f). The new system again features only five independent components of the covariant R-tensor; we observe that the third and fourth equations (i.e., equations 53c and 47d) contain the same quantity R(2,3,1,2).

The important fact, however, is that the alternate solution leads to R(1,2,1,2) in equation (53a), a quantity that does not appear in any of (47a-f); all the other contractions have already been produced by the main solution. Accordingly, (53a) replaces (47c) and thereby creates a combined system capable of resolving six independent components of the covariant R-tensor. We reorder the equations and present the final system as

$$R(1,2,1,2) + (k_1 k_2 - t_1^2)(K_{12}+D)/D = 0 \ , \tag{54a}$$

$$R(1,3,1,3) + (t_1 \gamma_1 - k_1 \gamma_2)(K_{13}-D')/D = 0 \ , \tag{54b}$$

$$R(2,3,2,3) + (k_2 \gamma_1 - t_1 \gamma_2)(K_{23}+D'')/D = 0 \ , \tag{54c}$$

$$R(1,3,1,2) + (k_1 k_2 - t_1^2)(K_{13}-D')/D = 0 \ , \tag{54d}$$

$$R(2,3,1,2) + (k_1 k_2 - t_1^2)(K_{23}+D'')/D = 0 \ , \tag{54e}$$

$$R(2,3,1,3) + (t_1 \gamma_1 - k_1 \gamma_2)(K_{23}+D'')/D = 0 \ . \tag{54f}$$

We remark that in (5.25) of [H], the quantity $R_{urst} \lambda^u \mu^r \lambda^s \mu^t$ is denoted C and is referred to as the Riemannian curvature of the section of space defined by λ, μ. This quantity is our R(1,2,1,2) from equation (54a). Similarly, R(1,3,1,3) and R(2,3,2,3) from (54b,c) can be called the Riemannian curvatures of the space sections defined by λ, ν, and by μ, ν, respectively.

5. APPLICATION TO HOTINE'S COORDINATES ω AND ϕ

5.1 Covariant Riemann-Christoffel Tensor

We specialize the parameters G and H to Hotine's coordinates ω and ϕ, and find out what it entails in terms of the covariant R-tensor and the space. The comparison of equations (10a-c) herein with the formulas (12.008) in [H] reveals the following correspondences:

$$f_1 = -\sin\omega , \qquad f_2 = \cos\omega , \qquad f_3 = 0 ;$$

$$g_1 = -\sin\phi \cos\omega , \qquad g_2 = -\sin\phi \sin\omega , \qquad g_3 = \cos\phi ;$$

$$h_1 = \cos\phi \cos\omega , \qquad h_2 = \cos\phi \sin\omega , \qquad h_3 = \sin\phi .$$

In consulting (12a-c'), we deduce that

$$T_{12} = \sin\phi , \qquad T_{13} = -\cos\phi , \qquad T_{23} = 0 . \tag{55}$$

$$T'_{12} = 0 , \qquad T'_{13} = 0 , \qquad T'_{23} = -1 , \tag{56}$$

which, together with (40a,b) and (49), yield

$$K_{12} = -\cos\phi , \qquad K_{13} = -\sin\phi , \qquad K_{23} = 0 . \tag{57}$$

Finally, from (34a-c) we have

$$D = \cos\phi , \qquad D' = -\sin\phi , \qquad D'' = 0 . \tag{58}$$

The result $D''=0$ indicates that G and H specialized for ω and ϕ make only the main solution and the alternate solution of the system (33) feasible, but not a third kind of solution.

From (57) and (58) we observe that

$$K_{12} + D = 0 , \qquad K_{13} - D' = 0 , \qquad K_{23} + D'' = 0 . \tag{59}$$

Thus, equations (54a-f) yield

$$R(1,2,1,2) = R(1,3,1,3,) = R(2,3,2,3)$$
$$= R(1,3,1,2) = R(2,3,1,2) = R(2,3,1,3) = 0 .$$

Upon employing the temporary coordinate system introduced below (47f), in which

$$\lambda^r = (1,0,0) , \qquad \mu^r = (0,1,0) , \qquad \nu^r = (0,0,1) ,$$

the above values of R(i,j,m,n) become R_{ijmn}. Accordingly, we have

$$R_{1212} = R_{1313} = R_{2323} = R_{1312} = R_{2312} = R_{2313} = 0 \ . \tag{60}$$

If the quantities in (60) represent six independent components of the covariant R-tensor, this tensor must be zero. Conversely, if all components of the covariant R-tensor are zero as a consequence of (60), the six components shown in (60) are independent. Due to the skew-symmetric properties of the covariant R-tensor, namely $R_{ijmn}=-R_{ijnm}$ and $R_{ijmn}=-R_{jimn}$, if the same index appears more than twice, the component is automatically zero. Thus, only the following permutations of indices could potentially result in nonzero components:

$$1212,\ 1213,\ 1223;\quad 1312,\ 1313,\ 1323;\quad 2312,\ 2313,\ 2323; \tag{61}$$

plus nine permutations (three in each group) with the last two indices interchanged, which only affects the sign; plus 18 permutations (six in each newly expanded group) with the first two indices interchanged, which again only affects the sign.

The above description reveals that if the nine permutations of indices in (61) result in zero components, all components must be zero. This is, indeed, the case here because for every permutation in (61) the corresponding component of the covariant R-tensor is forced to zero by one of the conditions in (60). Accordingly, this tensor must be zero. The components of the covariant R-tensor are not some general functions of ω and ϕ as one might expect from (54a-f), which would then happen to be zero at P, but, rather,

$$R_{ijmn} \equiv \text{constant} = 0 \ . \tag{62}$$

This is a direct consequence of the values in (59), which are identically zero regardless of ω and ϕ. The present argument could be repeated in conjunction with any N-surface; thus, (62) is valid regardless of ω, ϕ, and N, i.e., it is valid for any location. Since R_{ijmn} is identically zero in one coordinate system, the same is true in other systems. The tensor equation (62), which is our previous equation (8), reveals that the space must be flat. As a consequence, all orders of covariant derivatives of A_r, B_r, and C_r are zero, all Christoffel symbols in $\{x^r\}$ and all orders of their partial derivatives are zero, the locally Cartesian system at P is globally Cartesian, etc., as can be gathered from Appendix A.

5.2 Leg Derivatives

Leg derivatives and other quantities in Hotine's system can be formed upon substituting (55)-(57) and (62) into the appropriate relations. For example, the initial formulas (11a-c), specialized here as

$$\lambda_{rs} = (\sin\phi\ \mu_r - \cos\phi\ \nu_r)\omega_s ,$$

$$\mu_{rs} = -\sin\phi\ \lambda_r\ \omega_s - \nu_r\ \phi_s ,$$

$$\nu_{rs} = \cos\phi\ \lambda_r\ \omega_s + \mu_r\ \phi_s ,$$

are identical to equations (12.014-016) in [H]. As another example, the main solution provides the leg derivatives of ω and ϕ via (35) and (36), namely

$$\omega_{/1} = -k_1/\cos\phi , \qquad \omega_{/2} = -t_1/\cos\phi , \qquad \omega_{/3} = \gamma_1/\cos\phi . \qquad (63a)$$

$$\phi_{/1} = -t_1 , \qquad \phi_{/2} = -k_2 , \qquad \phi_{/3} = \gamma_2 , \qquad (63b)$$

implying that

$$\omega_s = (-k_1\lambda_s - t_1\mu_s + \gamma_1\nu_s)/\cos\phi , \qquad (63a')$$

$$\phi_s = -t_1\lambda_s - k_2\mu_s + \gamma_2\nu_s . \qquad (63b')$$

In the same way, the alternate solution yields via (48):

$$\omega_s = (\sigma_1\lambda_s + \sigma_2\mu_s + \varepsilon_3\nu_s)/\sin\phi ; \qquad (63a")$$

the alternate formula for ϕ_s is identical to (63b').

The above formulas (63a',b') are Hotine's equations (12.046,047). However, here they are obtained at an early stage, immediately following λ_{rs}, μ_{rs}, ν_{rs}, whereas in [H] the gradients ω_s and ϕ_s can be expressed only after individual components of λ^r, μ^r, ν^r, and λ_r, μ_r, ν_r have been found. We also comment that the alternate formula (63a") does not have an equivalent in [H]. On the other hand, (63a',b') and (63a") follow respectively from equations (A.4,5) and (8) in [Z]. The relations in our equation (37) express the curvatures σ_1, σ_2, and ε_3 in terms of the five curvature parameters; here they have the form

$$\sigma_1 = -k_1\tan\phi , \qquad \sigma_2 = -t_1\tan\phi , \qquad \varepsilon_3 = \gamma_1\tan\phi . \qquad (64)$$

Of these formulas, the first two correspond to (12.066,067) in [H], but the third is again without an equivalent there; however, it appears in [Z] as one of the equations in (9). We finally remark that we are henceforth entitled to use the relation

$$K = k_1 k_2 - t_1^2 ,$$

valid in the flat space, where K is the Gaussian curvature (see also equations D.17a-c in Appendix D).

The leg derivatives of curvatures in terms of the five curvature parameters can be readily transcribed from (42a-i) upon using the specializations (55)-(57) as above, and upon substituting zero for all of R(i,j,m,n) according to (62). The specialized formulas read

$$k_{1/2} - t_{1/1} = (K - k_1^2 - t_1^2)\tan\phi , \tag{65a}$$

$$k_{1/3} + \gamma_{1/1} = k_1^2 + t_1^2 + \gamma_1^2 - (k_1\gamma_2 - 2t_1\gamma_1)\tan\phi , \tag{65b}$$

$$t_{1/3} + \gamma_{1/2} = 2Ht_1 + \gamma_1\gamma_2 - [(k_1 - k_2)\gamma_1 + t_1\gamma_2]\tan\phi , \tag{65c}$$

$$k_{2/1} - t_{1/2} = 2Ht_1 \tan\phi , \tag{65d}$$

$$t_{1/3} + \gamma_{2/1} = 2Ht_1 + \gamma_1\gamma_2 + k_2\gamma_1 \tan\phi , \tag{65e}$$

$$k_{2/3} + \gamma_{2/2} = k_2^2 + t_1^2 + \gamma_2^2 - t_1\gamma_1 \tan\phi , \tag{65f}$$

$$\sigma_{1/2} - \sigma_{2/1} = K + (k_1^2 + t_1^2)\tan^2\phi , \tag{65g}$$

$$\varepsilon_{3/1} - \sigma_{1/3} = k_1\gamma_2 - t_1\gamma_1 + (k_1^2 + t_1^2 + \gamma_1^2)\tan\phi + t_1\gamma_1\tan^2\phi , \tag{65h}$$

$$\varepsilon_{3/2} - \sigma_{2/3} = -k_2\gamma_1 + t_1\gamma_2 + (2Ht_1 + \gamma_1\gamma_2)\tan\phi - k_1\gamma_1\tan^2\phi . \tag{65i}$$

In analogy to the comment made in the paragraph below (32'), the relations (65a) and (65d) correspond to Hotine's equations (8.23), the Mainardi-Codazzi equations, expressed now in terms of the five curvature parameters. The other relations in (65a-i) do not have equivalents in [H].

However, as a matter of verification, the formulas (65a-f) correspond respectively to the initial formulas for (ω_I), (ω_{II}), (ω_{III}), (ϕ_I), (ϕ_{II}), and (ϕ_{III}) appearing in the section "Hotine's Assertion" in [Z], where they are credited to Hotine (an unpublished report to the I.A.G. Toronto Assembly, 1957). And the formulas (65g-i) are equivalent to the version of (ω_I^*), (ω_{II}^*), and (ω_{III}^*) appearing below equation (10) in [Z], provided these quantities are combined with equation (9), [ibid.]. Since in [Z] and [H], as well as in

Hotine's unpublished report, the space is assumed to be flat from the outset, this agreement cross-validates the consistency of the pertinent derivations.

To obtain the differences of the double leg derivatives of ω from the main solution, we specialize (41a-c):

$$\omega_{/1/2} - \omega_{/2/1} = -(k_{1/2} - t_{1/1} - K\tan\phi)/\cos\phi \ . \tag{66a}$$

$$\omega_{/3/1} - \omega_{/1/3} = [k_{1/3} + \gamma_{1/1} + (k_1\gamma_2 - t_1\gamma_1)\tan\phi]/\cos\phi \ . \tag{66b}$$

$$\omega_{/2/3} - \omega_{/3/2} = -[t_{1/3} + \gamma_{1/2} + (t_1\gamma_2 - k_2\gamma_1)\tan\phi]/\cos\phi \ . \tag{66c}$$

In recalling the comment below (41c), we form analogous relations for ϕ:

$$\phi_{/1/2} - \phi_{/2/1} = k_{2/1} - t_{1/2} \ , \tag{66d}$$

$$\phi_{/3/1} - \phi_{/1/3} = t_{1/3} + \gamma_{2/1} \ , \tag{66e}$$

$$\phi_{/2/3} - \phi_{/3/2} = -k_{2/3} - \gamma_{2/2} \ . \tag{66f}$$

In turning to the alternate solution, from (51a-c) one has

$$\omega_{/1/2} - \omega_{/2/1} = (\sigma_{1/2} - \sigma_{2/1} - K)/\sin\phi \ , \tag{66a'}$$

$$\omega_{/3/1} - \omega_{/1/3} = (\varepsilon_{3/1} - \sigma_{1/3} - k_1\gamma_2 + t_1\gamma_1)/\sin\phi \ , \tag{66b'}$$

$$\omega_{/2/3} - \omega_{/3/2} = -(\varepsilon_{3/2} - \sigma_{2/3} + k_2\gamma_1 - t_1\gamma_2)/\sin\phi \ . \tag{66c'}$$

With regard to ϕ, the alternate solution gives an outcome identical to (66d-f). The differences of the double leg derivatives in (66a-f) and (66a'-c') do not have equivalents in [H] or [Z].

The differences of the double leg derivatives can be presented entirely in terms of the five curvature parameters. We first consider (66a-f), where we make appropriate substitutions from (65a-f). This results in

$$\omega_{/1/2} - \omega_{/2/1} = (k_1^2 + t_1^2)\tan\phi/\cos\phi \ , \tag{67a}$$

$$\omega_{/3/1} - \omega_{/1/3} = (k_1^2 + t_1^2 + \gamma_1^2 + t_1\gamma_1\tan\phi)/\cos\phi \ , \tag{67b}$$

$$\omega_{/2/3} - \omega_{/3/2} = -(2Ht_1 + \gamma_1\gamma_2 - k_1\gamma_1\tan\phi)/\cos\phi \ ; \tag{67c}$$

$$\phi_{/1/2} - \phi_{/2/1} = 2Ht_1\tan\phi \ , \tag{67d}$$

$$\phi_{/3/1} - \phi_{/1/3} = 2Ht_1 + \gamma_1\gamma_2 + k_2\gamma_1\tan\phi \ . \tag{67e}$$

$$\phi_{/2/3} - \phi_{/3/2} = -(k_2^2 + t_1^2 + \gamma_2^2 - t_1\gamma_1\tan\phi) \ . \tag{67f}$$

However, if we similarly make substitutions from (65g-i) into (66a'-c'), we recover the formulas (67a-c). Thus, equations (67a-f) uniquely express the differences of the double leg derivatives of Hotine's coordinates ω and ϕ in terms of the five curvature parameters. These formulas do not have equivalents in [H] or [Z].

5.3 Laplacians of ω and ϕ

In differentiating the basic gradient equation (9) covariantly and contracting (on both indices) with the associated metric tensor, we obtain ΔN, the Laplacian of N:

$$\Delta N = n_s\nu^s + ng^{rs}\nu_{rs} \ .$$

The first term on the right-hand side is $n_{/3}$, while $g^{rs}\nu_{rs}$ in the second term is $-2H$ according to Hotine's formula (7.19). It thus follows that

$$n_{/3} = \Delta N + 2Hn \equiv \Delta N + (k_1+k_2)n \ , \tag{68}$$

which is a standard result equivalent to Hotine's equation (12.100). Although this result is valid in a general space, here it is used for the flat space. Before expressing the Laplacians for ω and ϕ via equation (30), we need to formulate the invariants $\nu_{rst}\lambda^r g^{st}$ and $\nu_{rst}\mu^r g^{st}$, specialized for the flat space.

In the flat space, we have $N_{rst}=N_{str}=\ldots$, where any permutation of indices is permissible. It then follows that

$$(\Delta N)_r = g^{st}N_{str} = g^{st}N_{rst} = g^{st}(n_{st}\nu_r+n_s\nu_{rt}+n_t\nu_{rs}+n\nu_{rst}) \ ,$$

and, due to the symmetry of g^{st}, that

$$g^{st}\nu_{rst} = [(\Delta N)_r - \Delta n\nu_r - 2g^{st}n_s\nu_{rt}]/n \ . \tag{69}$$

If we contract (69) with λ^r and utilize (19b), we obtain

$$\nu_{rst}\lambda^r g^{st} = [(\Delta N)_{/1} + 2(k_1n_{/1}+t_1n_{/2}-\gamma_1n_{/3})]/n \ .$$

The definitions of γ_1 and γ_2 in (32') supply $n_{/1}$ and $n_{/2}$, while (68) supplies $n_{/3}$. Thus, the above equation leads to

$$\nu_{rst}\lambda^r g^{st} = 2(t_1\gamma_2 - k_2\gamma_1 - \gamma_1 \Delta N/n) + (1/n)(\Delta N)_{/1} \ . \tag{70a}$$

If we contract (69) with μ^r and utilize (19c), a similar procedure yields

$$\nu_{rst}\mu^r g^{st} = 2(t_1\gamma_1 - k_1\gamma_2 - \gamma_2 \Delta N/n) + (1/n)(\Delta N)_{/2} \ . \tag{70b}$$

To develop the Laplacian for ω, we first specialize (30):

$$\Delta\omega = \omega_{/1/1} + \omega_{/2/2} + \omega_{/3/3} - (\gamma_1 - \sigma_2)\omega_{/1} - (\gamma_2 + \sigma_1)\omega_{/2} - 2H\omega_{/3} \ . \tag{71}$$

In recalling (63a), we form

$$\omega_{/1/1} = (k_1 t_1 \tan\phi - k_{1/1})/\cos\phi \ , \qquad \omega_{/2/2} = (k_2 t_1 \tan\phi - t_{1/2})/\cos\phi \ ,$$

$$\omega_{/3/3} = (\gamma_1\gamma_2 \tan\phi + \gamma_{1/3})/\cos\phi \ , \tag{72}$$

where use has been made of (63b) as well. To determine $-k_{1/1}$, we adopt k_1 from the second alternative in (14a), perform the first-leg differentiation, and use the first formulas in (18a,c) together with (19b). This results in

$$-k_{1/1} = \nu_{rst}\lambda^r\lambda^s\lambda^t - 2t_1\sigma_1 + k_1\gamma_1 \ . \tag{73a}$$

Similarly, we adopt t_1 from the second alternative in (15), perform the second-leg differentiation, and use the second formulas in (18a,b,c) together with (19b). This yields

$$-t_{1/2} = \nu_{rst}\lambda^r\mu^s\mu^t - k_2\sigma_2 + k_1\sigma_2 + k_2\gamma_1 \ . \tag{73b}$$

Finally, we adopt γ_1 from the second alternative in (16a), perform the third-leg differentiation, and use the third formulas in (18a,c) together with (19b), which gives

$$\gamma_{1/3} = \nu_{rst}\lambda^r\nu^s\nu^t + \gamma_2\varepsilon_2 - k_1\gamma_1 - t_1\gamma_2 \ . \tag{73c}$$

If we substitute (73a-c) into (72), and substitute the new relations into (71), where also (63a) is to be used, we obtain an intermediate result

$$\Delta\omega = [\nu_{rst}\lambda^r g^{st} - t_1\sigma_1 - k_2\sigma_2 + \gamma_2\varepsilon_3 + (2Ht_1 + \gamma_1\gamma_2)\tan\phi]/\cos\phi \ ;$$

here advantage has been taken of the leg formulation of the associated metric tensor as seen prior to (29). From (64) it follows that the second, third, and fourth terms inside the brackets above form $(2Ht_1 + \gamma_1 \gamma_2)\tan\phi$. This new term, together with the expression (70a) substituted for the first term inside the brackets, transforms the intermediate result into

$$\Delta\omega = [-2(k_2\gamma_1 - t_1\gamma_2 + \gamma_1 \Delta N/n) + 2(2Ht_1 + \gamma_1\gamma_2)\tan\phi + (1/n)(\Delta N)_{/1}]/\cos\phi \ . \quad (74)$$

This Laplacian agrees with (12.104) of [H].

In the last step, (30) is specialized for ϕ to read

$$\Delta\phi = \phi_{/1/1} + \phi_{/2/2} + \phi_{/3/3} - (\gamma_1 - \sigma_2)\phi_{/1} - (\gamma_2 + \sigma_1)\phi_{/2} - 2H\phi_{/3} \ . \quad (75)$$

From (63b), the double leg derivatives readily follow as

$$\phi_{/1/1} = -t_{1/1} \ , \qquad \phi_{/2/2} = -k_{2/2} \ , \qquad \phi_{/3/3} = \gamma_{2/3} \ . \quad (76)$$

We adopt t_1 from the fourth alternative of (15), perform the first-leg differentiation, and use the first formulas in (18a,b,c) together with (19c); this yields

$$-t_{1/1} = \nu_{rst}\mu^r\lambda^s\lambda^t + k_1\sigma_1 - k_2\sigma_1 + k_1\gamma_2 \ . \quad (77a)$$

Next, we adopt k_2 from the second alternative in (14b), perform the second-leg differentiation, and use the second formulas in (18b,c) together with (19c), which gives

$$-k_{2/2} = \nu_{rst}\mu^r\mu^s\mu^t + 2t_1\sigma_2 + k_2\gamma_2 \ . \quad (77b)$$

Finally, we adopt γ_2 from the second alternative in (16b), perform the third-leg differentiation, and use the third formulas in (18b,c) together with (19c), which results in

$$\gamma_{2/3} = \nu_{rst}\mu^r\nu^s\nu^t - \gamma_1\varepsilon_3 - t_1\gamma_1 - k_2\gamma_2 \ . \quad (77c)$$

The substitution of (77a-c) and (63b) into (75) yields

$$\Delta\phi = \nu_{rst}\mu^r g^{st} + k_1\sigma_1 + t_1\sigma_2 - \gamma_1\varepsilon_3 \ .$$

A subsequent substitution by (70b) and (64) gives the Laplacian as

$$\Delta\phi = -2(k_1\gamma_2 - t_1\gamma_1 + \gamma_2\Delta N/n) - (k_1^2 + t_1^2 + \gamma_1^2)\tan\phi + (1/n)(\Delta N)_{/2} \ , \quad (78)$$

which agrees with (12.105) of [H].

6. SUMMARY AND CONCLUSION

To discuss the validity of Hotine's coordinates ω and ϕ, the latter have been treated in a generalized form G and H in a general Riemannian space. The conditions of symmetry of the second covariant derivatives of G and H result in six commutators containing the single and the double leg derivatives, i.e., the directional derivatives along the 3-leg λ, μ, ν, of G and H (see equations 28a-c applied to both G and H). The formulation of the commutators entails the task of expressing pairwise combinations (sums or differences) of the leg derivatives of eight curvatures, which brings forth the covariant Riemann-Christoffel tensor, called here the covariant R-tensor, contracted with certain permutations of contravariant leg vectors (see equations 31a-i). With the aid of G and H, combinations of the leg derivatives can also be expressed in terms of the five curvature parameters k_1, k_2, t_1, γ_1, and γ_2 (see equations 42a-i). In either formulation, the combinations of the leg derivatives feature the contracted covariant R-tensor, and are valid in a general space.

Accordingly, also the six commutators feature the contracted covariant R-tensor and the five curvature parameters. In addition, the commutators feature some quite general functions of G and H, which uniquely relate the 3-leg to the Cartesian axes A, B, C of a locally Cartesian coordinate system $\{x^r\}$ at the point P. From the definition of $\{x^r\}$, it holds true, in the flat space as well as in curved spaces, that

$$A_{rs} = B_{rs} = C_{rs} = 0 \ .$$

These tensor equations apply at P, as do all the other relations in this study; the only exceptions occur when a tensor, such as the metric tensor, is expanded in the Taylor series from P to arbitrary locations.

If the characteristics of the space, and thereby the components of the covariant R-tensor, are known, the commutators represent conditions on the above-mentioned general functions of G and H. On the other hand, if such functions are known or chosen, the commutators represent conditions on the covariant R-tensor and thereby on the space. The latter possibility forms the backbone of the present analysis. In essence, it enables one to choose the coordinates, and to require that the space conform to this choice. Since the covariant R-tensor has six independent components, six distinct contractions must be constrained by the commutators in order to make a complete resolution of this tensor feasible. Upon relating the partial derivatives of G and H (with respect to the coordinates $\{x^r\}$) to the curvature parameters in one of three possible systems of linear equations, only five independent contractions are obtained, the sixth being repetitious. Another system must then be resolved, supplying the sixth independent constraint. The general analysis in terms of G and H culminates in equations (54a-f), which are equivalent to a linearly independent system of six G- and H-commutators featuring the five curvature parameters, capable of resolving the six independent components of the covariant R-tensor.

The resolution of six independent components, and thereby of all components, of the covariant R-tensor can be readily accomplished in a temporary coordinate system adopted as locally Cartesian along the 3 leg (belonging to P). The results can then be transformed to any other coordinate system. In general, the components of the covariant R-tensor at points on a given N-surface may be some functions of G and H, which, at P, could turn out to have the zero values. The curvature tensor would then be zero not only in the temporary coordinate system but in any other system, such as $\{x^r\}$. This would entail

$$A_{rst} = B_{rst} = C_{rst} = 0 \, , \tag{79}$$

implying that the local system $\{x^r\}$ is Cartesian to at least a second order. Equivalently, the relation

$$k_{rst} = k_{rts} \tag{79'}$$

would hold true for a general vector k belonging to P. However, if the partial derivatives (with respect to $\{x^r\}$ as usual) of the covariant R-tensor departed from zero, it would follow that $A_{rstu} \neq 0$, etc., and the local system would be Cartesian only to a second order. Based on the components of the covariant R-tensor and on the components' various partial derivatives at P, one could construct, via the Taylor-series expansion, the metric tensor in $\{x^r\}$ for arbitrary points, and thereby concretely express the space.

If six independent components of the covariant R-tensor are identically zero (regardless of G, H and N), i.e., if

$$R_{ijmn} \equiv \text{constant} = 0 \tag{80}$$

holds true for these components and thus for any components of the covariant R-tensor, then the following relations are valid:

$$A_{rst} = B_{rst} = C_{rst} = 0 , \qquad A_{rstu} = B_{rstu} = C_{rstu} = 0, \; \ldots \; . \tag{81}$$

where the dots represent all the other sets of covariant derivatives. Equivalently, one can state that the local system is globally Cartesian (or Cartesian to any order), which, in turn, implies that the space must be flat. The solution (80) characterizes the specialization of G and H to Hotine's coordinates ω and ϕ. That is to say, if one chooses ω and ϕ as the first two coordinates and examines what kind of space this choice entails, the answer is that the only admissible space is the flat space. In any other space, equation (80) would not hold true, and ω, ϕ would be inadmissible as coordinates even in a small neighborhood of the point P.

In view of the above, the answer to the question posed in the Introduction is that ω and ϕ may exist as coordinates only in the flat space. Thus, the outcome of the feasibility study concerned with the admissibility of the (ω, ϕ, N) coordinate system in a curved space is negative. The flatness of the space is reflected in that all orders of covariant derivatives of A_r, B_r, C_r are zero Not only must (79) hold true, as well as all it implicitly entails, but the additional relations in (81) must also hold true, as well as all they entail. There is no need to enforce the flatness separately via conditions of the kind (79'), which are now satisfied as a by-product. The Mainardi-Codazzi equations, for example, would represent even a weaker condition than (79') because they correspond to the specialization $\nu_{rst} = \nu_{rts}$. These and other such equations are then regarded merely as identities in the flat space, where, however, they may be of great value in their own right. In conclusion, the admissibility of Hotine's (ω, ϕ, N) coordinate system has been restricted to the flat space (or its regions). In the process, a number of equations from [Z], applicable to the flat space, have been cross-validated by independent means, and several new relations have been presented that do not have equivalents in [H] or [Z].

APPENDIX A

LOCALLY CARTESIAN COORDINATE SYSTEM

A.1 Metric Tensor

We consider a general Riemannian space (flat or curved), where, at a given point P, we establish a locally Cartesian coordinate system as discussed in §5-6 of [Hotine, 1969]. This system, denoted $\{x^r\}$, r=1,2,3, has the property that the Christoffel symbols (the C-symbols) at P are zero for any indices i, j, k:

$$\Gamma^i_{jk} = 0 \ . \tag{A.1}$$

When written in a matrix form characterized by brackets, at P the metric tensor g_{rs} and the associated metric tensor g^{rs} are

$$[g_{rs}] = [g^{rs}] = I \ . \tag{A.2}$$

The components of an orthonormal triad A, B, and C identifying the coordinate lines at P are given by

$$A_r = A^r = (1,0,0) \ , \qquad B_r = B^r = (0,1,0) \ , \qquad C_r = C^r = (0,0,1) \ . \tag{A.3}$$

Since g_{rs} is constant under the covariant differentiation, it follows that

$$\partial g_{rs}/\partial x^t = \Gamma^i_{rt} g_{is} + \Gamma^i_{st} g_{ri} \ ,$$

from which the ordinary partial derivatives yield $\partial^2 g_{rs}/\partial x^t \partial x^k$, $\partial^3 g_{rs}/\partial x^t \partial x^k \partial x^u$, etc.

Due to (A.1), for P we deduce

$$\partial g_{rs}/\partial x^t = 0 \ , \tag{A.4a}$$

$$\partial^2 g_{rs}/\partial x^t \partial x^k = \partial(\Gamma^s_{rt} + \Gamma^r_{st})/\partial x^k \ , \tag{A.4b}$$

$$\partial^3 g_{rs}/\partial x^t \partial x^k \partial x^u = \partial^2(\Gamma^s_{rt} + \Gamma^r_{st})/\partial x^k \partial x^u \ , \tag{A.4c}$$

etc., where we have also utilized (A.2), as well as (A.4a) in the subsequent expressions. An n-th order ($n>3$) partial derivative of g_{rs} comprises an (n-1)-th order partial derivative of the C-symbols in parentheses above, plus terms containing products of (n-3)-th, (n-4)-th, etc., down to the first-order partial derivatives of the C-symbols; there are no (n-2)-th order partial derivatives of these symbols present. We are now in a position to formulate the metric tensor g'_{rs} at a point P', where the coordinate differences from P are Δx^t, by means of the Taylor series:

$$g'_{rs} = g_{rs} + (1/2)[\partial(\Gamma^s_{rt} + \Gamma^r_{st})/\partial x^k]\Delta x^t \Delta x^k + (1/6)[\partial^2(\Gamma^s_{rt} + \Gamma^r_{st})/\partial x^k \partial x^u]\Delta x^t \Delta x^k \Delta x^u + \dots \qquad (A.5)$$

This is not a tensor equation, but a relation expressing individual components of the metric tensor at P' from individual components of this tensor at P. As in other cases, the former can be computed from the latter via the Taylor series. One readily ascertains that such an expansion cannot produce a tensor equation. At the outset, g'_{rs} on the left-hand side of (A.5) is associated with P', whereas g_{rs} on the right-hand side is associated with P; in a tensor equation all quantities would be associated with the same point. Furthermore, the terms beyond the first on the right-hand side are not tensors at all.

If the expression inside the first brackets of (A.5) is nonzero, the system $\{x^r\}$ is Cartesian to a first order, and as such, is confined to P and its immediate neighborhood. If this expression is zero but the expression inside the next brackets is nonzero, the system is Cartesian to a second order, etc. If all orders of partial derivatives of the C-symbols are zero at P, we have $g'_{rs}=g_{rs}$ (equality for components, not a tensor equation) for any point P', and the system is globally Cartesian. Clearly, the converse is also true; in a system that is Cartesian throughout the space, the metric tensor (A.2) is constant everywhere, making the C-symbols and all of their partial derivatives identically zero.

A.2 Riemann-Christoffel Tensors

We now turn our attention to the Riemann-Christoffel tensor and its covariant version, which we call respectively the R-tensor and the covariant R-tensor. The former is derived in §5-3 of [Hotine, 1969], and is presented as

$$R^u_{.ijk} = \partial\Gamma^u_{ik}/\partial x^j - \partial\Gamma^u_{ij}/\partial x^k + \Gamma^m_{ik}\Gamma^u_{mj} - \Gamma^m_{ij}\Gamma^u_{mk} \, . \qquad (A.6)$$

Due to (A.1), in the system $\{x^r\}$ at P we have

$$\partial\Gamma^u_{ik}/\partial x^j = \partial\Gamma^u_{ij}/\partial x^k + R^u_{.ijk} \, . \qquad (A.7)$$

The R-tensor is linked to the covariant R-tensor by

$$R^{u}_{.ijk} = g^{um} R_{mijk} .$$

At the point P, all components of the two tensors are equal in the system $\{x^r\}$ because of (A.2). Except for possible sign differences, due to its symmetric and skew-symmetric properties the covariant R-tensor has only six distinct components (in three dimensions). This fact is exploited in the body of the present study.

To relate the partial derivatives of the two kinds of the R-tensors in analogy to (A.4a-c) we first deduce the partial derivatives of g^{rs} at P:

$$\partial g^{rs}/\partial x^{t} = 0 , \tag{A.8a}$$

$$\partial^{2} g^{rs}/\partial x^{t}\partial x^{k} = -\partial(\Gamma^{r}_{ts} + \Gamma^{s}_{tr})/\partial x^{k} , \tag{A.8b}$$

$$\partial^{3} g^{rs}/\partial x^{t}\partial x^{k}\partial x^{u} = -\partial^{2}(\Gamma^{r}_{ts} + \Gamma^{s}_{tr})/\partial x^{k}\partial x^{u} , \tag{A.8c}$$

etc. Although the right-hand sides of (A.8b,c) are equal to the negative of the right-hand sides of (A.4b,c), such a relationship does not exist beyond the third-order partial derivatives. However, the pattern for the n-th order partial derivatives is the same as that described following (A.4c). Numerical values of individual components of the two R-tensors and their partial derivatives are related here by

$$R^{u}_{.ijk} = R_{uijk} , \tag{A.9a}$$

$$\partial R^{u}_{.ijk}/\partial x^{t} = \partial R_{uijk}/\partial x^{t} , \tag{A.9b}$$

$$\partial^{2} R^{u}_{.ijk}/\partial x^{t}\partial x^{p} = \partial^{2} R_{uijk}/\partial x^{t}\partial x^{p} - R_{mijk}\partial(\Gamma^{u}_{tm} + \Gamma^{m}_{tu})/\partial x^{p} , \tag{A.9c}$$

etc., where (A.8b) has been utilized in (A.9c). (In the latter, the summation convention for the index m applies regardless of its position.) An n-th order ($n>1$) partial derivative of $R^{u}_{.ijk}$ can be shown to comprise an n-th order partial derivative of R_{uijk}, plus terms containing products of (n-2)-th, (n-3)-th, etc., down to 0-th order partial derivatives of the covariant R-tensor with the first-, second-, etc., up to (n-1)-th order partial derivatives of the C-symbols. The pattern shows a certain symmetry. For example, if n=4, these terms comprise products of second-order partial derivatives (or 2-derivatives) of the covariant R-tensor with 1-derivatives of the C-symbols, 1-derivatives of the covariant R-tensor with 2-derivatives of the C-symbols, and 0-derivatives of the covariant R-tensor (i.e., the tensor itself) with 3-derivatives of the C-symbols. The above equations, as all equations in this appendix with the exception of (A.5), apply in the locally Cartesian system $\{x^r\}$ at P.

If all components of the covariant R-tensor are zero, i.e., if

$$R_{uijk} = 0 \ , \tag{A.10}$$

so are the components of the R-tensor (see the formula below equation A.7). Although this is true in all coordinate systems, we utilize it only for $\{x^r\}$. Equation (A.7) indicates that in such a case it is permissible to adopt

$$\partial\Gamma^u_{ik}/\partial x^j = 0 \ , \tag{A.11}$$

where the indices u, i, k, and j are unrestricted. From (A.5) we observe that the local system is now Cartesian to at least a second order. The strategy of adopting a choice of the kind (A.11) stems from the fact that if all orders of partial derivatives of the C-symbols are admissible to be zero, then the system $\{x^r\}$ is admissible to be globally Cartesian and the space must be flat.

On the other hand, *if (A.10) is not valid* for all components, neither is (A.11), and the system is Cartesian only to a first order. We can build the metric tensor g'_{rs} in (A.5) by computing the values of the partial derivatives of the C-symbols. In this process, we conveniently set to zero those values that are not constrained otherwise. In particular, due to the properties of the two R-tensors, all components of these tensors in (A.9a) are zero if j=k and/or u=i. Accordingly, (A.7) allows us to set

$$\partial\Gamma^u_{ik}/\partial x^k = 0 \ , \qquad \partial\Gamma^i_{ik}/\partial x^j = 0 \ , \tag{A.12}$$

where the indices are unrestricted (here the repeating indices do not entail the summation convention). As an alternative, one could keep the first equation in (A.12) intact but, instead of the second equation, adopt $\partial\Gamma^i_{kj}/\partial x^i = 0$. This would yield $\partial\Gamma^i_{ik}/\partial x^j = R^i_{.kji}$ as opposed to the second equation in (A.12). Upon using (A.12) and the symmetry property of the C-symbols, the remaining elements of these symbols at P can be found from (A.7) and then utilized in (A.5).

To express higher-order derivatives of the C-symbols with respect to the coordinates $\{x^r\}$, we differentiate (A.6) and specialize it for P:

$$\partial^2\Gamma^u_{ik}/\partial x^j\partial x^t = \partial^2\Gamma^u_{ij}/\partial x^k\partial x^t + \partial R^u_{.ijk}/\partial x^t \ . \tag{A.13}$$

In working systematically with the covariant R-tensor, we can replace the second term on the right-hand side by its equivalent from (A.9b). In paralleling (A.10-11), we then state that if it holds true for all elements that

$$\partial R_{uijk}/\partial x^t = 0 \ , \tag{A.14}$$

it is admissible to adopt

$$\partial^2\Gamma^u_{ik}/\partial x^j\partial x^t = 0 \ , \tag{A.15}$$

where the indices are unrestricted. This expression can then be substituted into (A.5). It follows that if both (A.10) and (A.14) are valid, the system $\{x^r\}$ is Cartesian to at least a third order. On the other hand, even when (A.14) does not apply, it is still true that $\partial R^u_{.ijk}/\partial x^t=0$ if $j=k$ and/or $u=i$. According to (A.13), in this case it is admissible to set

$$\partial^2\Gamma^u_{ik}/\partial x^k\partial x^t = 0 \ , \qquad \partial^2\Gamma^i_{ik}/\partial x^j\partial x^t = 0 \ , \tag{A.16}$$

which is in a close analogy to (A.12). The remaining elements of the doubly differentiated C-symbols can be found from (A.13) and utilized in (A.5).

In differentiating (A.6) twice and specializing it for P, one has

$$\begin{aligned}\partial^3\Gamma^u_{ik}/\partial x^j\partial x^t\partial x^p &= \partial^3\Gamma^u_{ij}/\partial x^k\partial x^t\partial x^p + \partial^2 R^u_{.ijk}/\partial x^t\partial x^p \\ &\quad - (\partial\Gamma^m_{ik}/\partial x^t)(\partial\Gamma^u_{mj}/\partial x^p) - (\partial\Gamma^m_{ik}/\partial x^p)(\partial\Gamma^u_{mj}/\partial x^t) \\ &\quad + (\partial\Gamma^m_{ij}/\partial x^t)(\partial\Gamma^u_{mk}/\partial x^p) + (\partial\Gamma^m_{ij}/\partial x^p)(\partial\Gamma^u_{mk}/\partial x^t) \ .\end{aligned} \tag{A.17}$$

An n-th order $(n>2)$ partial derivative of the C-symbol can be shown to contain another n-th order partial derivative of the C-symbol, an $(n-1)$-th order partial derivative of the R-tensor, plus terms containing products of $(n-2)$-th, $(n-3)$-th, etc., down to the first-order partial derivatives of the C-symbols in a somewhat symmetric manner. For example, if $n=5$, the latter terms contain products of 3-derivatives with 1-derivatives, and of 2-derivatives with 2-derivatives (of the C-symbols).

Working again $(n>2)$ in terms of the covariant R-tensor and considering, for example, $n=3$ and thus (A.17), we can replace the second term on the right-hand side by its equivalent from (A.9c) involving compatible products of the C-symbol derivatives. If, in addition to (A.10) making (A.11) admissible, we also have

$$\partial^2 R^u_{.ijk}/\partial x^t \partial x^p = 0 \ , \tag{A.18}$$

it is further admissible to set

$$\partial^3 \Gamma^u_{ik}/\partial x^j \partial x^t \partial x^p = 0 \ ,$$

which we can then substitute into (A.5). Consequently, if all of (A.10), (A.14), and (A.18) are valid, the system $\{x^r\}$ is Cartesian to at least a fourth order as can be confirmed upon consulting the last paragraph of Section A.1. If (A.18) does not apply, the consideration of j=k and/or u=i can be used for choices analogous to (A.16), but only if the first-order partial derivatives of the C-symbols presented in (A.17) are zero. However, such a restriction does not curtail the possibility to compute the third-order partial derivatives of the C-symbols. When compared with the cases where n<3, the present procedure may simply have to resort to (A.17) for additional elements for which the zero values would produce a conflict. After the triply differentiated C-symbols have been formed, they can be utilized in (A.5) in accordance with the description below (A.4c).

The foregoing contains all the information necessary to proceed to any order of partial derivatives of the C-symbols, and to include any number of terms in the Taylor-series expansion of the metric tensor in (A.5). The entire development is based on the covariant R-tensor. If it holds true for all of its elements (or, equivalently, for its six independent elements) that

$$R_{uijk} \equiv \text{constant} = 0 \ , \tag{A.19}$$

i.e., that *all* of (A.10), (A.14), (A.18), etc., at the point P apply, then all orders of partial derivatives of the C-symbols at P are admissible to be zero, the system $\{x^r\}$ is admissible to be globally Cartesian, and the space must be flat. (Consistent with this statement, equation A.5 yields $[g_{rs}]\equiv$constant=I.) It is readily apparent that the converse is also true. Equation (A.19) and its consequences just stated are crucial elements in the present study.

As a parenthetical note, we state that the zero value of the covariant R-tensor at P, as well as the zero values of all its spatial derivatives there, are equivalent to this tensor being zero at all points of the space. Clearly, if the covariant R-tensor and all its spatial derivatives are zero at P, then this tensor is zero at any point of the space, as per its Taylor-series expansion from P. The converse is also true. For, suppose that the covariant

R-tensor is zero at P as well as at every point of the space, and that the (zero) value of this tensor at every point is expressed in the Taylor series around the (zero) value at P. Due to the independence of the individual terms of the series, and to the series being identically zero, the linear term for any point must be zero, the quadratic term for any point must be zero, etc. But since the linear term equals the partial derivative on the left-hand side of (A.14) contracted with Δx^t, and this term must be zero for every Δx^t, it follows that the first-order partial derivative in (A.14) must be zero. A similar outcome is noted for the (symmetric) partial derivatives of higher orders.

In returning to (A.19), we state that if this equation does not apply, the system $\{x^r\}$ is Cartesian to a certain order. As has been demonstrated, if, at P, the partial derivatives of the covariant R-tensor are zero up to and including the n-th order (n=0 corresponds to the original tensor), but not beyond this order, the system is Cartesian to an (n+2)-th order. We have means at our disposal to express the metric tensor in $\{x^r\}$ at an arbitrary point P'. In particular, in using the values of the covariant R-tensor and its partial derivatives at P, we can form the R-tensor and its partial derivatives at P. This, in turn, leads to an evaluation of the partial derivatives of the C-symbols at P (where the symbols themselves are zero), and, with the aid of the Taylor-series expansion, to an evaluation of the metric tensor at arbitrary locations. We have thus developed concrete means by which to describe the space, based on the locally Cartesian coordinate system $\{x^r\}$ at the point P.

A.3 Cartesian Triad

In Section A.1, we have presented the orthonormal triad A, B, C at the point P, oriented along the coordinates lines of the locally Cartesian coordinate system $\{x^r\}$. The components of the Cartesian triad have been listed in (A.3). The covariant differentiation of A_r yields

$$A_{rs} = \partial A_r / \partial x^s - \Gamma^i_{rs} A_i ,$$

which, by virtue of the definition of A_r in (A.3) and of the system $\{x^r\}$, is zero. A similar outcome is reached for B_{rs} and C_{rs}, so that we have

$$A_{rs} = B_{rs} = C_{rs} = 0 . \tag{A.20}$$

This is a tensor equation at P, valid in any coordinates. Here it will be used only in the coordinate system $\{x^r\}$. Equation (A.20) has nothing to do with the curvature of the space, which does not affect the first covariant derivatives of vectors (and the second covariant derivatives of scalars).

If we differentiate A_{rs}, B_{rs}, and C_{rs} covariantly in $\{x^r\}$, at P we have

$$A_{rst} = -\partial\Gamma^1_{rs}/\partial x^t \ , \qquad B_{rst} = -\partial\Gamma^2_{rs}/\partial x^t \ , \qquad C_{rst} = -\partial\Gamma^3_{rs}/\partial x^t \ , \tag{A.21}$$

which shows that A_{rst}, B_{rst}, and C_{rst} are symmetric in the first two indices. If the system $\{x^r\}$ is Cartesian to at least a second order, i.e., if

$$\partial\Gamma^i_{rs}/\partial x^t = 0 \ , \tag{A.22}$$

where the indices are unrestricted, it follows that

$$A_{rst} = B_{rst} = C_{rst} = 0 \ . \tag{A.23}$$

Conversely, if (A.23) holds true, then (A.21) is equivalent to (A.22) and the system is Cartesian to at least a second order. We note that (A.23) appears as (2) in the body of the present study.

In employing the covariant R-tensor, we can make a useful deduction related to (A.23). We begin by assuming the following tensor equations, considered at P in the coordinate system $\{x^r\}$:

$$A_{rst} = A_{rts} \ , \qquad B_{rst} = B_{rts} \ , \qquad C_{rst} = C_{rts} \ . \tag{A.24}$$

The first of these equations is expressed here as

$$A_{rst} - A_{rts} = R^u{}_{.rst}A^u = R_{urst}A^u = R^1{}_{.rst} = R_{1rst} = 0 \ ,$$

which holds true for unrestricted r, s, and t. The same argument repeated for B and C shows that

$$R_{irst} = 0 \ , \tag{A.25}$$

where also i is unrestricted. In recalling the path from (A.10) to (A.11), we deduce that (A.22) is admissible. But this outcome has already resulted above in (A.23), a tensor equation valid in any coordinates. We have thus established that in spite of (A.24) appearing weaker than (A.23), the two equations are equivalent.

The system $\{x^r\}$ would be Cartesian to at least a third order if it also held true that

$$\partial^2\Gamma^i_{rs}/\partial x^t \partial x^p = 0 \ . \tag{A.26}$$

In differentiating covariantly A_r, B_r, and C_r three times in succession and considering the results to apply at P as usual, we have

$$A_{rstu} = -\partial^2\Gamma^1_{rs}/\partial x^t \partial x^u \ , \qquad B_{rstu} = -\partial^2\Gamma^2_{rs}/\partial x^t \partial x^u \ ,$$

$$C_{rstu} = -\partial^2\Gamma^3_{rs}/\partial x^t \partial x^u \ ,$$

where only the definition of $\{x^r\}$, implying (A.1) and (A.20), has been used. Equation (A.26) is thus seen to be equivalent to

$$A_{rstu} = B_{rstu} = C_{rstu} = 0 \ , \tag{A.27}$$

which is listed, in a similar context, as equation (3).

An n-th order (n>3) covariant derivative of A_r comprises the minus (n-1)-th order partial derivative of Γ^1_{rs}, plus terms containing products of (n-3)-th, (n-4)-th, etc., down to the first-order partial derivatives of the C-symbols. Thus, as (A.5) indicates, if the system is Cartesian to an n-th order, it entails a relation similar to (A.27) but corresponding to an n-fold covariant differentiation. Conversely, if such a relation, as well as relations of this kind expressing all of the lower orders of covariant differentiation, hold true, then the system is Cartesian to an n-th order. In summary, if k is the highest order of covariant differentiation that has not yet breached the validity of a relation of the type (A.27), then all orders in the range 1 through k-1 of partial derivatives of the C-symbols produce zeros as admissible results, and the system $\{x^r\}$ is Cartesian to a k-th order.

To justify the statement of equivalence made in the Introduction in conjunction with equations (2) and (4), we consider an arbitrary vector k at the point P, and perform

$$k_{rst} - k_{rts} = R^u_{.rst} k_u = R_{urst} k^u \ .$$

The vector k can be expressed by means of A, B, and C as

$$k^u = aA^u + bB^u + cC^u \ ,$$

which, when substituted in the above equation, yields

$$k_{rst} - k_{rts} = aR_{1rst} + bR_{2rst} + cR_{3rst} \, , \tag{A.28}$$

where a, b, and c are arbitrary scalar invariants (including arbitrary constants). If $\{x^r\}$ is Cartesian to at least a second order so that (A.25) applies, it follows that

$$k_{rst} = k_{rts} \, . \tag{A.29}$$

Conversely, if (A.29) holds true, then, due to the arbitrariness of a, b, and c, equation (A.28) yields (A.25) and thereby also (A.23). We remark that although (A.29) implies that the covariant R-tensor is zero at P, it has no connection to the partial derivatives of this tensor. The above represents one approach to establishing the equivalence of the tensor equations (A.29) and (A.23).

We can also begin with the tensor equation

$$k_r = aA_r + bB_r + cC_r \, ,$$

differentiate it covariantly twice, interchange the second and the third indices, and form

$$k_{rst} - k_{rts} = a(A_{rst} - A_{rts}) + b(B_{rst} - B_{rts}) + c(C_{rst} - C_{rts}) \, . \tag{A.30}$$

This result would have been obtained even without taking advantage of (A.20), and regardless of whether a, b, and c are arbitrary scalar invariants (functions of position) or arbitrary constants; in either case we have $a_{st}=a_{ts}$, $b_{st}=b_{ts}$, $c_{st}=c_{ts}$ (in the latter case these equations would become the identities 0=0). If we make the substitution seen below (A.24) with respect to A, and similarly with respect to B and C, we recover (A.28). However, in working directly with (A.30), we observe that if $\{x^r\}$ is Cartesian to at least a second order, equation (A.29) follows. Conversely, if (A.29) holds true, equation (A.30) indicates that (A.24) and thereby (A.23) must be true as well because of the arbitrariness of a, b, and c.

We have thus shown the equivalence of tensor equations (A.29) and (A.23). In conclusion, (A.29) does not imply that the space is flat, only that the system $\{x^r\}$ can be Cartesian to at least a second order. If (A.27) is not satisfied, $\{x^r\}$ is Cartesian only to a second order. If it is satisfied, as well as all the other relations of its type (for covariant differentiation of any order), then $\{x^r\}$ can be globally Cartesian and the space must be flat.

APPENDIX B

TAYLOR-SERIES EXPANSION OF METRIC TENSOR IN POLAR COORDINATES

In a general three-dimensional space, the metric tensor in the coordinates symbolized by $\{x^r\}$, $r=1,2,3$, can be expanded in the Taylor series as follows:

$$g'_{rs} = g_{rs} + (\partial g_{rs}/\partial x^p)\Delta x^p + (1/2)(\partial^2 g_{rs}/\partial x^p \partial x^q)\Delta x^p \Delta x^q + \ldots \quad \text{(B.1)}$$

Here g_{rs} is the metric tensor at a point P and g'_{rs} is the metric tensor at a point P'; further, $\Delta x^r \equiv x'^r - x^r$, where x^r symbolizes the coordinates of P and x'^r symbolizes the coordinates of P'. In accordance with §3-12 in [Hotine, 1969], we have the tensor equation

$$g_{rs,p} \equiv \partial g_{rs}/\partial x^p - \Gamma^t_{rp} g_{ts} - \Gamma^t_{sp} g_{tr} = 0 .$$

This yields

$$\partial g_{rs}/\partial x^p = \Gamma^t_{rp} g_{ts} + \Gamma^t_{sp} g_{tr} , \quad \text{(B.2a)}$$

from which it follows that

$$\partial^2 g_{rs}/\partial x^p \partial x^q = \partial(\Gamma^t_{rp} g_{ts} + \Gamma^t_{sp} g_{tr})/\partial x^q , \ldots . \quad \text{(B.2b)}$$

The Taylor expansion is formalized by substituting (B.2a,b) into (B.1).

In a two-dimensional space, where the coordinate system is symbolized by $\{u^\alpha\}$, $\alpha=1,2$, the above outcome is rewritten as

$$a'_{\alpha\beta} = a_{\alpha\beta} + (\Gamma^\omega_{\alpha\gamma} a_{\omega\beta} + \Gamma^\omega_{\beta\gamma} a_{\omega\alpha})\Delta u^\gamma + (1/2)[\partial(\Gamma^\omega_{\alpha\gamma} a_{\omega\beta} + \Gamma^\omega_{\beta\gamma} a_{\omega\alpha})/\partial u^\delta]\Delta u^\gamma \Delta u^\delta + \ldots . \quad \text{(B.3)}$$

If we now stipulate that the two-dimensional space is flat, i.e., a plane, and define the coordinates as r (distance from the origin) and θ (angle from a given axis x), i.e., as polar coordinates, we have $\{u^\alpha\} \equiv (r, \theta)$ and

$$ds^2 = dr^2 + r^2 d\theta^2 . \quad \text{(B.4)}$$

The metric (B.4) implies the following metric tensor (in matrix notation):

$$[a_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} . \quad \text{(B.5)}$$

Equation (B.5) can also be deduced via a transformation of coordinates between the systems $\{\bar{u}^{\alpha}\} \equiv (x, y)$ and $\{u^{\alpha}\} \equiv (r, \theta)$. In particular, the transformation formula

$$a_{\alpha\beta} = (\partial\bar{u}^{\gamma}/\partial u^{\alpha})(\partial\bar{u}^{\delta}/\partial u^{\beta})\bar{a}_{\gamma\delta} \ ,$$

if applied to overbarred coordinates as Cartesian x, y, yields

$$[a_{\alpha\beta}] = [\partial\bar{u}^{\gamma}/\partial u^{\alpha}]^{T}[\partial\bar{u}^{\gamma}/\partial u^{\beta}] \ .$$

Since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$[\partial\bar{u}^{\gamma}/\partial u^{\alpha}] = \begin{bmatrix} x/r & -y \\ y/r & x \end{bmatrix} ,$$

and (B.5) follows, for a given point, upon using $x^2 + y^2 = r^2$.

To express the Christoffel symbols (the C-symbols), we use the notational convention of §3-1 in [Hotine, 1969]. Thus, we write

$$\Gamma^{\mu}_{\alpha\beta} = a^{\mu\gamma}[\alpha\beta, \gamma] \ , \qquad \text{(B.6)}$$

where

$$[\alpha\beta, \gamma] = (1/2)(\partial a_{\beta\gamma}/\partial u^{\alpha} + \partial a_{\alpha\gamma}/\partial u^{\beta} - \partial a_{\alpha\beta}/\partial u^{\gamma}) \ . \qquad \text{(B.7)}$$

Since $a_{\alpha\mu}a^{\mu\gamma} = \delta^{\gamma}_{\alpha}$, i.e., $[a_{\alpha\beta}]$ and $[a^{\alpha\beta}]$ are inverses of each other, in reference to (B.5) one has

$$[a^{\mu\gamma}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix} . \qquad \text{(B.8)}$$

Equation (B.7) in conjunction with (B.5) yields, upon arranging the entries for α and β in $[\alpha\beta, \gamma]$ into a matrix for a given γ:

$$[\alpha\beta, 1] = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} , \qquad [\alpha\beta, 2] = \begin{bmatrix} 0 & r \\ r & 0 \end{bmatrix} .$$

Similarly, for a given μ, the C-symbols $\Gamma^{\mu}_{\alpha\beta}$ are presented as

$$\Gamma^{1}_{\alpha\beta} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} , \qquad \Gamma^{2}_{\alpha\beta} = \begin{bmatrix} 0 & 1/r \\ 1/r & 0 \end{bmatrix} .$$

(In this paragraph we refrain from using brackets on the left-hand sides, normally symbolizing matrix notation.) Since

$$\Gamma^{\omega}_{\alpha 1} a_{\omega\beta} = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}, \qquad \Gamma^{\omega}_{\alpha 2} a_{\omega\beta} = \begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix},$$

and similarly for $\Gamma^{\omega}_{\beta 1} a_{\omega\alpha}$ (the interchange of α and β entails a transposition), we have

$$\Gamma^{\omega}_{\alpha\gamma} a_{\omega\beta} + \Gamma^{\omega}_{\beta\gamma} a_{\omega\alpha} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix} \ldots \text{ for } \gamma = 1, \quad = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ldots \text{ for } \gamma = 2 .$$

Accordingly,

$$(1/2)\partial(\Gamma^{\omega}_{\alpha\gamma} a_{\omega\beta} + \Gamma^{\omega}_{\beta\gamma} a_{\omega\alpha})/\partial u^{\delta} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \ldots \text{ for } \gamma = \delta = 1 ,$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ldots \text{ in the other three cases.}$$

With the above development, (B.3) yields

$$[a'_{\alpha\beta}] = [a_{\alpha\beta}] + \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}\Delta r + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Delta\theta + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\Delta r^2 + 0\Delta r\Delta\theta + 0\Delta\theta\Delta r + 0\Delta\theta^2 ,$$

where the third- and higher-order terms are zero in this example. We then have

$$[a'_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2r\Delta r \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Delta r^2 \end{bmatrix} . \tag{B.9}$$

But this is

$$[a'_{\alpha\beta}] = \begin{bmatrix} 1 & 0 \\ 0 & r'^2 \end{bmatrix} , \tag{B.9'}$$

where

$$r' = r + \Delta r .$$

Thus, (B.9) corresponds to (B.9'), where r'^2 is developed via the Taylor-series expansion as

$$r'^2 = r^2 + 2r\Delta r + \Delta r^2 .$$

The present simple example illustrates that individual components of $a'_{\alpha\beta}$ may be developed via the Taylor series. The component results may be grouped together in a convenient form, such as (B.3). From their construction, the terms on the right-hand side of (B.3), except for the first, are not tensors but expressions of a given order in coordinate differences. Clearly, the convenient grouping of components that gives rise to such expressions cannot in itself create tensors. However, the fact that (B.3) is not a tensor equation does not detract from its usefulness.

APPENDIX C

DETAILED DERIVATION OF EIGHT CURVATURES

The eight curvatures listed in Section 3.1 will now be derived in detail. These curvatures consist of five curvature parameters k_1, k_2, t_1, γ_1, and γ_2, and three additional curvatures σ_1, σ_2, and ε_1; as stated in Section 3.1, they correspond to the orthonormal triad λ, μ, ν. In conjunction with a more general orthonormal triad ℓ, j, ν, the eight curvatures are denoted, in the same order, as k, k*, t, $\bar{\gamma}_1$, $\bar{\gamma}_2$, σ, σ^*, and $\bar{\varepsilon}_3$. Except for the three barred quantities newly introduced, the notation is adopted from [Hotine, 1969], abbreviated as [H]. Although some of the latter eight curvatures are derived in Chapter 7 of [H] (see equations 7.04, 7.03, and 7.08 for σ, k, and t, respectively), here we begin with a complete development associated with the general triad ℓ, j, ν, and subsequently transcribe the results for the triad λ, μ, ν. This will help in making the present study as self-contained as practicable.

C.1 Development Associated with the General Triad

The extrinsic properties of surface curves are developed upon considering such curves in both the space and the surface context. At a given point P, the unit tangent vector to a surface curve is denoted ℓ, and the orthonormal surface vector is denoted j. As in §7-1 of [H], ℓ and j are also considered to be unit tangents to a family of surface curves and to a family of their orthogonal trajectories, respectively. This interpretation allows us to differentiate ℓ_α or j_α (or other tensors defined along a line) covariantly with respect to surface coordinates; subsequently, the application of such a differentiation can be restricted to a particular line at a particular point. As is explained in §4-1 of [H], similar considerations regarding families of lines are valid in three dimensions.

With ν denoting the unit normal to the surface at P, we define ℓ, j, ν as a right-handed orthonormal triad. This triad is called "general", since ℓ and j are not restricted to follow any particular surface directions. The development will not be affected by the curvature of the space because no double covariant differentiation (in space) will take place, and, therefore, the same formulas will be obtained whether the space is flat or curved.

The present geometrical situation is depicted in Fig. 1 below, which is essentially Fig. 6 in [H]. In addition to the vectors ℓ, j, and ν, the figure shows the unit vector **m**, the principal normal to the curve whose unit tangent is ℓ, and the unit vector n, the binormal. Similar to ℓ, j, ν, the vectors ℓ, **m**, n also form a right-handed orthonormal triad. Except for ℓ, pointing into the plane of the paper, the remaining four vectors lie in the plane of the paper.

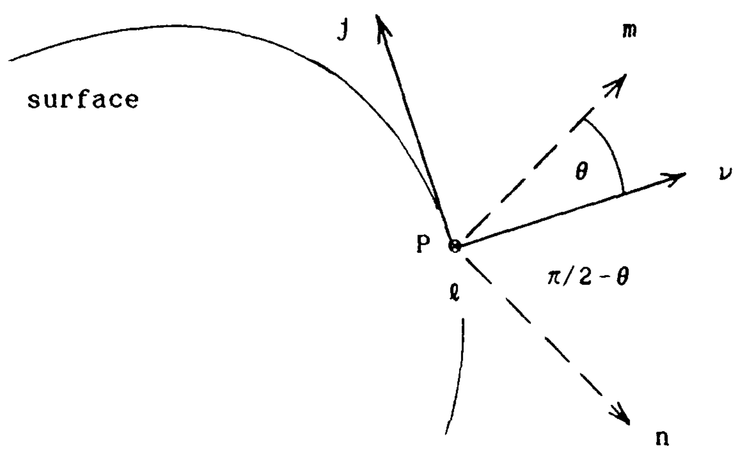

Fig. 1

The <u>first part</u> of the development follows essentially §7-1 through §7-4 in [H]. The preliminary formulas needed for this part are listed below. In the surface context, we have

$$\ell_{\alpha\beta} = \sigma j_\alpha \ell_\beta + \sigma^* j_\alpha j_\beta \ , \tag{C.1}$$

which is the first equation in (4.11) of [H]. This readily yields

$$\ell_{\alpha\beta}\ell^\beta = \sigma j_\alpha \ , \qquad \ell_{\alpha\beta} j^\alpha \ell^\beta = \sigma \ , \tag{C.2a,b}$$

where (C.2a) is (4.07) in [H]. In the space context, (C.2a) corresponds to

$$\ell_{rs}\ell^s = \chi m_r \ , \tag{C.3}$$

which is the first of the three equations in (4.06) of [H], known as the Frenet equations. The quantities χ and σ are respectively the principal and the geodesic curvatures of the curve; in analogy to other quantities associated with j and distinguished by an asterisk, σ^* represents the geodesic curvature of the curve's orthogonal trajectory (in the direction j).

We will also need

$$\partial x^r/\partial u^\alpha \equiv x^r_\alpha = \ell^r \ell_\alpha + j^r j_\alpha \ , \tag{C.4}$$

which corresponds to (6.02) and (6.09) in [H], except that here the surface coordinates are symbolized by $\{u^\alpha\}$ instead of $\{x^\alpha\}$. Finally, (6.16) in [H] gives

$$x^r_{\alpha\beta} = b_{\alpha\beta}\nu^r \ ; \tag{C.5}$$

the symmetric surface tensor $b_{\alpha\beta}$ is known as the second fundamental form of the surface. We note that all of the tensor equations (C.1-5) apply at the point P. Thus, all of the tensors in these equations (including the mixed tensors x^r_α and $x^r_{\alpha\beta}$ as well as the scalar invariants σ, σ^*, and χ) belong to P.

We begin the first part of the development with (6.07) in [H]:

$$\ell^r = x^r_\alpha \ell^\alpha \ ,$$

whose surface covariant derivative with respect to u^β *is*

$$\ell^r_s x^s_\beta = x^r_{\alpha\beta}\ell^\alpha + x^r_\alpha \ell^\alpha_\beta \ ;$$

ℓ^r_s and ℓ^α_β could also be written as $\ell^r{}_{,s}$ and $\ell^\alpha{}_{,\beta}$, respectively. Upon using (C.5), (C.4), and (C.1) with the index α raised, it follows that

$$\ell^r_s x^s_\beta = \nu^r(b_{\alpha\beta}\ell^\alpha) + j^r(\sigma\ell_\beta + \sigma^* j_\beta) \ ,$$

which is (7.01) in [H]. The contraction with ℓ^β (where, on the left-hand side, Hotine's formula 6.07 is again used), the lowering of the space index r, and the utilization of (C.3) result in

$$\ell_{rs}\ell^s = \chi m_r = \nu_r(b_{\alpha\beta}\ell^\alpha\ell^\beta) + \sigma j_r \ , \tag{C.6}$$

corresponding closely to (7.02) in [H]. This equation confirms that the vector m lies in the plane containing j and ν.

From Fig.1 it follows that

$$m_r\nu^r = \cos\theta \ , \qquad m_r j^r = \sin\theta \ . \tag{C.7a,b}$$

Furthermore, we define an invariant k as

$$k = \chi\cos\theta \ . \tag{C.8}$$

Finally, we recall from (3.19,20) in [H] that if p and q are two orthonormal vectors, it holds true that

$$p_{rs}p^r = 0 \ , \qquad p_{rs}q^r = -q_{rs}p^r \ , \tag{C.9a,b}$$

which applies also in two dimensions. If we now contract (C.6) with ν^r and use (C.7a,8,9b), we obtain

$$k = \chi\cos\theta = \ell_{rs}\nu^r\ell^s = -\nu_{rs}\ell^r\ell^s = b_{\alpha\beta}\ell^\alpha\ell^\beta \ , \tag{C.10}$$

which corresponds to (7.03) in [H]. Further, if we contract (C.6) with j^r and use (C.7b,9b) as well as (C.2b), we find

$$\sigma = \chi\sin\theta = \ell_{rs}j^r\ell^s = -j_{rs}\ell^r\ell^s = \ell_{\alpha\beta}j^\alpha\ell^\beta = -j_{\alpha\beta}\ell^\alpha\ell^\beta \ , \tag{C.11}$$

which corresponds to (7.04) and the subsequent equation in [H]. Due to (C.9a), the contraction of (C.6) with ℓ^r merely yields the identity 0=0.

Equation (C.10) shows that k is the same for any surface curve in the direction ℓ, since $b_{\alpha\beta}\ell^\alpha\ell^\beta$ depends only on $b_{\alpha\beta}$ (a point function) and the direction ℓ, not on any particular curve in this direction. In §7-03 of [H], this quantity is identified as the normal curvature of the surface in the direction ℓ. In reference to (C.11), if

$$\sigma = 0 \ , \tag{C.12a}$$

i.e., if the curve is a geodesic, in general ($\chi \neq 0$) it follows that

$$\theta = 0 \ , \tag{C.12b}$$

and, according to Fig. 1 and equation (C.10), also that

$$m \equiv \nu \ , \qquad k = \chi \ . \tag{C.12c,d}$$

In this case, the principal normal coincides with the surface normal, and the normal curvature of the surface in ℓ equals the space curvature in ℓ, as is explained in §7-04 of [H]. In general, the first equalities in (C.10,11) relate the curvatures χ, k, and σ by

$$\chi^2 = k^2 + \sigma^2 \ ; \tag{C.13}$$

this equation could be used to define the normal curvature through

$$k^2 = \chi^2 - \sigma^2 .$$

The <u>second part</u> of the present development parallels the first part, except that the direction considered is j instead of ℓ. Accordingly, the pertinent quantities are attributed an asterisk to distinguish them from their counterparts dealt with previously. This applies, specifically, to the curvatures σ, χ, and k (along ℓ) being replaced by σ^*, χ^*, and k^* (along j), respectively, as well as to the vectors **m** and n being replaced by **m*** and n*, and to the angle θ being replaced by θ^*. The vector ℓ is replaced by j while the vector j is replaced by $-\ell$, since the positive rotation from j is toward $-\ell$. This situation is depicted in Fig. 2, whose construction is conceptually similar to Fig. 1.

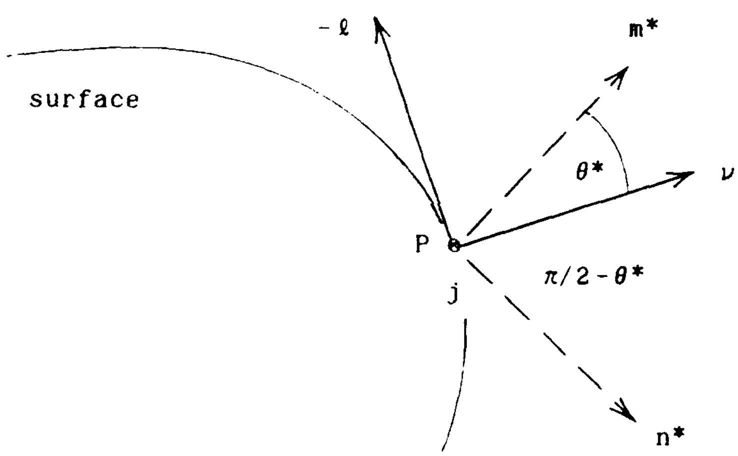

Fig. 2

Instead of (C.1-2b) as preliminary formulas, we now use the second equation in (4.11) of [H], and write

$$j_{\alpha\beta} = -\sigma\ell_\alpha\ell_\beta - \sigma^*\ell_\alpha j_\beta ; \tag{C.14}$$

$$j_{\alpha\beta}j^\beta = -\sigma^*\ell_\alpha , \qquad j_{\alpha\beta}\ell^\alpha j^\beta = -\sigma^* . \tag{C.15a,b}$$

In analogy to (C.3), we have

$$j_{rs}j^s = \chi^* m^*_r , \tag{C.16}$$

while (C.4,5) need no modification. Similarly to the first part, we begin the development with

$$j^r = x^r_\alpha j^\alpha ,$$

which, when differentiated covariantly with respect to u^β, yields

$$j^r_s x^s_\beta = x^r_{\alpha\beta} j^\alpha + x^r_\alpha j^\alpha_\beta .$$

Upon using (C.5), (C.4), and (C.14) with the index α raised, it follows that

$$j^r_s x^s_\beta = \nu^r (b_{\alpha\beta} j^\alpha) - \ell^r (\sigma \ell_\beta + \sigma^* j_\beta) .$$

The contraction with j^β, the lowering of the space index r, and the utilization of (C.16) result in

$$j_{rs} j^s = \chi^* m^*_r = \nu_r (b_{\alpha\beta} j^\alpha j^\beta) - \sigma^* \ell_r . \tag{C.17}$$

This equation parallels (C.6), but is without an equivalent in [H].

With the aid of Fig. 2, the expressions paralleling (C.7a,b) are seen to be

$$m^*_r \nu^r = \cos\theta^* , \qquad m^*_r \ell^r = -\sin\theta^* ; \tag{C.18a,b}$$

the expression paralleling (C.8) is

$$k^* = \chi^* \cos\theta^* . \tag{C.19}$$

To obtain relations paralleling (C.10,11), we contract (C.17) in turn with ν^r and $-\ell^r$, and use (C.18a-19) as well as (C.9b) and (C.15b):

$$k^* = \chi^* \cos\theta^* = j_{rs} \nu^r j^s = -\nu_{rs} j^r j^s = b_{\alpha\beta} j^\alpha j^\beta , \tag{C.20}$$

$$\sigma^* = \chi^* \sin\theta^* = -j_{rs} \ell^r j^s = \ell_{rs} j^r j^s = -j_{\alpha\beta} \ell^\alpha j^\beta = \ell_{\alpha\beta} j^\alpha j^\beta ; \tag{C.21}$$

the contraction of (C.17) with j^r would yield the identity 0=0. Here k^* again depicts a property of the surface, this time in the direction j, and, in analogy to (C.12a-d), the geodesic in this direction is characterized by

$$\sigma^* = 0 , \qquad \theta^* = 0 , \tag{C.22a,b}$$

$$m^* \equiv \nu , \qquad k^* = \chi^* . \tag{C.22c,d}$$

In paralleling (C.13), we now have $\chi^{*2} = k^{*2} + \sigma^{*2}$.

In the <u>third part</u> of the development, we return to the curve in the direction ℓ and recall the third equation from (4.06) in [H]:

$$n_{rs}\ell^s = -\tau m_r \ , \tag{C.23}$$

where τ is the torsion of the curve. From Fig. 1 we deduce that

$$n_r = \nu_r \sin\theta - j_r \cos\theta \ ,$$

which, when differentiated covariantly with respect to the arc element ds of the curve (interpreted in a usual manner as a member of a family of curves), yields

$$\begin{aligned} n_{rs}\ell^s = -\tau m_r = \nu_{rs}\ell^s \sin\theta - j_{rs}\ell^s \cos\theta \\ + (\nu_r \cos\theta + j_r \sin\theta) d\theta/ds \ , \end{aligned} \tag{C.24}$$

where (C.23) has been incorporated. Upon contracting this equation with j^r, and considering (C.7b) and (C.9a,b), it follows that

$$(\tau + d\theta/ds)\sin\theta = -\nu_{rs} j^r \ell^s \sin\theta = j_{rs}\nu^r \ell^s \sin\theta \ .$$

With the exception of $\sin\theta$ being replaced by $\cos\theta$, this relation is obtained upon contracting (C.24) with ν^r and considering also (C.7a). Thus, for any θ, we have

$$\tau + d\theta/ds = j_{rs}\nu^r \ell^s = -\nu_{rs} j^r \ell^s \ , \tag{C.25}$$

which is (7.05) in [H]. Since for a geodesic it holds true that $\theta=0$ and hence $d\theta/ds=0$, the right-hand side of (C.25) gives the torsion of a geodesic.

As a matter of interest, (C.25) applied to a geodesic can be derived separately as follows. First, (C.24) is contracted with m^r:

$$n_{rs} m^r \ell^s = -\tau = \nu_{rs} m^r \ell^s \sin\theta - j_{rs} m^r \ell^s \cos\theta + (\cos^2\theta + \sin^2\theta) d\theta/ds \ ,$$

or

$$\tau + d\theta/ds = j_{rs} m^r \ell^s \cos\theta - \nu_{rs} m^r \ell^s \sin\theta \ .$$

For a geodesic, (C.12c) implies that

$$m \equiv \nu \ ,$$

and (C.12b) implies that

$$\theta = 0 \ , \qquad d\theta/ds = 0 \ ,$$

as has already been stated below (C.25). We thus have

$$\tau = j_{rs} \nu^r \ell^s \ ,$$

confirming (C.25) for a geodesic.

To derive another form of $\tau + d\theta/ds$, we rewrite (6.19) of [H]:

$$-\nu_{rs} x^r_\alpha x^s_\beta = b_{\alpha\beta} \ ,$$

which we contract in turn with $j^\alpha \ell^\beta$ and $\ell^\alpha j^\beta$. In using the formula of the type (6.07) in [H], we obtain

$$-\nu_{rs} j^r \ell^s = -\nu_{rs} \ell^r j^s = b_{\alpha\beta} j^\alpha \ell^\beta = b_{\alpha\beta} \ell^\alpha j^\beta \ , \tag{C.26}$$

where the last equality justifies the others; this equality stems from the symmetry of $b_{\alpha\beta}$. The quantity $b_{\alpha\beta} \ell^\alpha j^\beta$, which depends on the point function $b_{\alpha\beta}$ and the direction ℓ but not on any particular curve in this direction, is denoted t in §7-5 of [H], and identified as the geodesic torsion of the surface in the direction ℓ. In collecting the expressions in (C.25,26), which all determine the same quantity $\tau + d\theta/ds$, and apply (C.9b) in one additional instance, we write

$$\begin{aligned} t = \tau + d\theta/ds &= j_{rs} \nu^r \ell^s = -\nu_{rs} j^r \ell^s = \ell_{rs} \nu^r j^s = -\nu_{rs} \ell^r j^s \\ &= b_{\alpha\beta} \ell^\alpha j^\beta = b_{\alpha\beta} j^\alpha \ell^\beta \ . \end{aligned} \tag{C.27}$$

This relation corresponds to (7.08) in [H].

The <u>fourth part</u> of the development parallels the third part, except that the direction considered is j instead of ℓ. We have seen a similar distinction between the second and the first parts. Here again, the explicit development presented below is not contained in [H]. In analogy to (C.23), we now have

$$n^*_{rs} j^s = -\tau^* m^*_r \ , \tag{C.28}$$

while from Fig. 2 we deduce that

$$n^*_r = \nu_r \sin\theta^* + \ell_r \cos\theta^* \ .$$

whose covariant differentiation with respect to the arc element ds* (along j) yields

$$n^*_{rs} j^s = -\tau^* m^*_r = \nu_{rs} j^s \sin\theta^* + \ell_{rs} j^s \cos\theta^*$$
$$+ (\nu_r \cos\theta^* - \ell_r \sin\theta^*) d\theta^*/ds^* . \qquad (C.29)$$

In contracting (C.29) in turn with ℓ^r and ν^r, and considering (C.18a,b) and (C.9a,b), we obtain

$$\tau^* + d\theta^*/ds^* = -\ell_{rs}\nu^r j^s = \nu_{rs}\ell^r j^s , \qquad (C.30)$$

multiplied in the former case by $\sin\theta^*$, and in the latter case by $\cos\theta^*$. Accordingly, this relation holds true for any θ^*. (Similarly to the discussion below equation C.25, upon contracting C.29 with m^{*r} it can be showed separately that C.30 is valid for a geodesic.) From the first equality in (C.30), we observe that $\tau^*+d\theta^*/ds^*$ is the negative of $\tau+d\theta/ds$ in (C.27). In analogy to its counterpart, $\tau^*+d\theta^*/ds^*$ depends on the direction j but not on any particular curve in that direction; it is denoted t* and identified as the geodesic torsion of the surface in the direction j. Thus, we have

$$t^* = -t . \qquad (C.31)$$

As stated in §7-6 of [H], the sum of the geodesic torsions in any two perpendicular directions is zero.

In the fifth part, we follow essentially §12-17 in [H], but consider the more general directions ℓ and j instead of λ and μ. As in [H], the starting point is provided by the notion of N-surfaces defined by N=constant. Thus, the gradient vector at a given point is perpendicular to the pertinent N-surface, and can be written as

$$N_r = n\nu_r . \qquad (C.32)$$

where n is the magnitude of N_r (not to be confused with the binormal n dealt with previously), and ν is a unit normal to the N-surface (earlier referred to simply as surface). The covariant differentiation of (C.32) yields

$$N_{rs} = n_s\nu_r + n\nu_{rs} ,$$

where the tensor N_{rs} is symmetric in r and s. If we interchange r and s, and subtract, we obtain (12.018) of [H]. The subsequent contraction with ν^s gives

$$n\nu_{rs}\nu^s = n_r - (n_s\nu^s)\nu_r ,$$

which is (12.019) in [H]. However, in our general context, n_r is expressed as

$$n_r = (n_s\ell^s)\ell_r + (n_s j^s)j_r + (n_s\nu^s)\nu_r ,$$

resulting in

$$n\nu_{rs}\nu^s = (n_s\ell^s)\ell_r + (n_s j^s)j_r .$$

Upon the division by n, the last equation becomes

$$\nu_{rs}\nu^s = \bar\gamma_1\ell_r + \bar\gamma_2 j_r , \qquad (C.33)$$

where

$$\bar\gamma_1 = (n_s\ell^s)/n \equiv (1/n)\partial n/\partial s ,$$

$$\bar\gamma_2 = (n_s j^s)/n \equiv (1/n)\partial n/\partial s^* ,$$

ds and ds* being length elements in the directions ℓ and j, respectively. If we contract (C.33) in turn with ℓ^r and j^r, and use (C.9b), we can express $\bar\gamma_1$ and $\bar\gamma_2$ more completely:

$$\bar\gamma_1 = (n_s\ell^s)/n = \nu_{rs}\ell^r\nu^s = -\ell_{rs}\nu^r\nu^s , \qquad (C.34a)$$

$$\bar\gamma_2 = (n_s j^s)/n = \nu_{rs}j^r\nu^s = -j_{rs}\nu^r\nu^s . \qquad (C.34b)$$

We remark that if γ is the principal curvature in the direction ν, and w is the principal normal, (C.33) can be restated as

$$\gamma w_r = \bar\gamma_1\ell_r + \bar\gamma_2 j_r . \qquad (C.35)$$

Consistent with §12-17 in [H], the principal normal is seen to be an N-surface vector. In writing (C.35) with the index r raised, and contracting the two equations, we also have

$$\gamma^2 = (\bar\gamma_1)^2 + (\bar\gamma_2)^2 .$$

Finally, in the sixth part we give the motivation for defining a quantity called $\bar\varepsilon_3$. If we seek, for example, the intrinsic derivatives of ℓ (the unit tangent to an N-surface curve) with respect to length elements in the ℓ, j, and ν directions, i.e., if we seek $\ell_{rs}\ell^s$, $\ell_{rs}j^s$, and $\ell_{rs}\nu^s$, we arrive at expressions where all of the needed tensor invariants have already been formed except for one, denoted $\bar{\varepsilon}_3$:

$$\bar{\varepsilon}_3 = \ell_{rs} j^r \nu^s = -j_{rs} \ell^r \nu^s . \tag{C.36}$$

This can be seen as follows. In considering a general expression of the form

$$h_r = (h_t \ell^t) \ell_r + (h_t j^t) j_r + (h_t \nu^t) \nu_r ,$$

where h_r is a covariant vector or the gradient of a scalar, in conjunction with (C.9a) we have

$$\ell_{rs} \ell^s = (\ell_{ts} \ell^s j^t) j_r + (\ell_{ts} \ell^s \nu^t) \nu_r = \sigma j_r + k \nu_r ,$$

$$\ell_{rs} j^s = (\ell_{ts} j^s j^t) j_r + (\ell_{ts} j^s \nu^t) \nu_r = \sigma^* j_r + t \nu_r ,$$

$$\ell_{rs} \nu^s = (\ell_{ts} \nu^s j^t) j_r + (\ell_{ts} \nu^s \nu^t) \nu_r = \bar{\varepsilon}_3 j_r - \bar{\gamma}_1 \nu_r ,$$

where, in the first line, we have used (C.11) and (C.10) giving σ and k, respectively, and in the second line we have used (C.21) and (C.27) giving σ^* and t, respectively. The third line brings forth the new expression in (C.36), as well as $\bar{\gamma}_1$ from (C.34a).

C.2 Transcription for the Triad λ, μ, ν

As has been indicated at the outset, the final step of the present development consists in transcribing the main outcome of the preceding section from a general orthonormal triad ℓ, j, ν to the orthonormal triad λ, μ, ν. The curvatures to be transcribed have been derived in the order: k in (C.10), σ in (C.11), k* in (C.20), σ^* in (C.21), t in (C.27), $\bar{\gamma}_1$ in (C.34a), $\bar{\gamma}_2$ in (C.34b), and $\bar{\varepsilon}_3$ in (C.36). Since $t^* = -t$ according to (C.31), no transcription is needed for t*. The transcribed curvatures will be presented in the order just listed, where the expressions not containing the orthonormal vectors will be omitted. The three overbarred quantities will have the overbar removed; of the other curvatures, those associated with the direction λ (replacing ℓ) will be attributed the subscript 1, and those associated with the direction μ (replacing j) will be attributed the subscript 2.

Below we list the transcribed curvatures in terms of the orthonormal vectors λ, μ, and ν.

$$k_1 = \lambda_{rs}\nu^r\lambda^s = -\nu_{rs}\lambda^r\lambda^s = b_{\alpha\beta}\lambda^\alpha\lambda^\beta \ , \tag{C.37}$$

$$\sigma_1 = \lambda_{rs}\mu^r\lambda^s = -\mu_{rs}\lambda^r\lambda^s = \lambda_{\alpha\beta}\mu^\alpha\lambda^\beta = -\mu_{\alpha\beta}\lambda^\alpha\lambda^\beta \ , \tag{C.38}$$

$$k_2 = \mu_{rs}\nu^r\mu^s = -\nu_{rs}\mu^r\mu^s = b_{\alpha\beta}\mu^\alpha\mu^\beta \ , \tag{C.39}$$

$$\sigma_2 = -\mu_{rs}\lambda^r\mu^s = \lambda_{rs}\mu^r\mu^s = -\mu_{\alpha\beta}\lambda^\alpha\mu^\beta = \lambda_{\alpha\beta}\mu^\alpha\mu^\beta \ , \tag{C.40}$$

$$\begin{aligned} t_1 &= \mu_{rs}\nu^r\lambda^s = -\nu_{rs}\mu^r\lambda^s = \lambda_{rs}\nu^r\mu^s = -\nu_{rs}\lambda^r\mu^s \\ &= b_{\alpha\beta}\lambda^\alpha\mu^\beta = b_{\alpha\beta}\mu^\alpha\lambda^\beta \ , \end{aligned} \tag{C.41}$$

$$\gamma_1 = \nu_{rs}\lambda^r\nu^s = -\lambda_{rs}\nu^r\nu^s \ , \tag{C.42}$$

$$\gamma_2 = \nu_{rs}\mu^r\nu^s = -\mu_{rs}\nu^r\nu^s \ , \tag{C.43}$$

$$\varepsilon_3 = \lambda_{rs}\mu^r\nu^s = -\mu_{rs}\lambda^r\nu^s \ . \tag{C.44}$$

As has been indicated, there is no need to list t_2. Since $t_2 = -t_1$, this quantity would give rise to no new contractions.

APPENDIX D

DETAILED DERIVATION OF LEG DERIVATIVES OF CURVATURES

D.1 Leg Derivatives of Curvatures Developed Using Space Vectors

In this section, equations (31a-i) are derived using the orthonormal vectors λ, μ, and ν in the space context. The initial equations serving in this task are (13a)-(17). The covariant R-tensor and its properties play a crucial role here. In accordance with §5-4 and §5-5 in [Hotine, 1969], abbreviated as [H], this tensor is skew-symmetric in the first two and the last two (space) indices, and symmetric with respect to the two pairs of indices:

$$R_{urst} = -R_{rust}\ , \qquad R_{urst} = -R_{urts}\ , \qquad R_{urst} = R_{stur}\ . \qquad \text{(D.1a,b,c)}$$

From (D.1a-c), or directly from (5.08) in [H], it follows that

$$R_{urst} = R_{tsru}\ . \qquad \text{(D.1d)}$$

The notation used in Section 4.1 for the leg derivatives, such as

$$(k_1)_t \mu^t \equiv k_{1/2}\ , \qquad \text{(D.2)}$$

etc., and for contractions of the covariant R-tensor, such as

$$R_{urst}\lambda^u \nu^r \lambda^s \mu^t \equiv R(1,3,1,2)\ , \qquad \text{(D.3)}$$

etc., will continue to be used throughout; there is no need to list the longer expressions on the left-hand sides of (D.2,3). Accordingly, (D.1a-d) yield

$$R(1,3,1,2) = -R(3,1,1,2)\ , \qquad R(1,3,1,2) = -R(1,3,2,1)\ , \qquad \text{(D.4a,b)}$$

$$R(1,3,1,2) = R(1,2,1,3)\ , \qquad R(1,3,1,2) = R(2,1,3,1)\ , \qquad \text{(D.4c,d)}$$

indicating that the numbers inside the parentheses can be manipulated similar to the indices of the covariant R-tensor.

Further, due to the fact that

$$R^u_{.rst}\lambda_u = R_{urst}\lambda^u\ ,$$

we can write

$$R^u_{.rst}\lambda_u \nu^r \lambda^s \mu^t = R(1,3,1,2)\ .$$

However, since according to (5.02) in [H] one has

$$\lambda_{rst} - \lambda_{rts} = R^{u}_{.rst}\lambda_{u} ,$$

it follows that

$$(\lambda_{rst} - \lambda_{rts})\nu^{r}\lambda^{s}\mu^{t} = R(1,3,1,2) , \tag{D.5}$$

i.e., the numbers inside the parentheses attributed to R reflect the ranking of the orthonormal vectors in the complete expression on the left-hand side, provided the contracting indices are properly ordered. Finally, the results in the present development are formalized with the aid of (18a)-(19c).

We are now in a position to derive (31a-i) in a direct and expeditious manner. From the first alternatives in (14a) and (15), we form

$$k_{1/2} = \lambda_{rst}\nu^{r}\lambda^{s}\mu^{t} + \lambda_{rs}\nu^{r}_{t}\lambda^{s}\mu^{t} + \lambda_{rs}\nu^{r}\lambda^{s}_{t}\mu^{t} ,$$

$$t_{1/1} = \lambda_{rst}\nu^{r}\mu^{s}\lambda^{t} + \lambda_{rs}\nu^{r}_{t}\mu^{s}\lambda^{t} + \lambda_{rs}\nu^{r}\mu^{s}_{t}\lambda^{t} ,$$

where expressions of the type (D.2) have been utilized. (Such manipulations will no longer be referenced.) Due to the first formula in (18a) and the second formula in (18c), symbolized as (18a)-1 and (18c)-2, the second term giving $k_{1/2}$ is $-k_2\sigma_1$; while due to (19b) and (18a)-2, the third term is $t_1\sigma_2 - t_1\gamma_1$. Due to (18a)-2 and (18c)-1, the second term giving $t_{1/1}$ is $-t_1\sigma_2$; while due to (19b) and (18b)-1, the third term is $-k_1\sigma_1 - t_1\gamma_1$. Next, the (dummy) indices s and t in the first term giving $t_{1/1}$ are interchanged, and the entire equation is subtracted from that giving $k_{1/2}$. Upon considering (D.5), the first term of the resulting equation is R(1,3,1,2); the other terms are obtained by subtracting $-t_1\sigma_2 - k_1\sigma_1 - t_1\gamma_1$ from $-k_2\sigma_1 + t_1\sigma_2 - t_1\gamma_1$ as indicated above. Accordingly, we have

$$k_{1/2} - t_{1/1} = R(1,3,1,2) + (k_1 - k_2)\sigma_1 + 2t_1\sigma_2 , \tag{D.6}$$

which is (31a).

Similarly, from the first alternatives in (14a) and (16a), we form

$$k_{1/3} = \lambda_{rst}\nu^{r}\lambda^{s}\nu^{t} + \lambda_{rs}\nu^{r}_{t}\lambda^{s}\nu^{t} + \lambda_{rs}\nu^{r}\lambda^{s}_{t}\nu^{t} ,$$

$$\gamma_{1/1} = -\lambda_{rst}\nu^{r}\nu^{s}\lambda^{t} - \lambda_{rs}\nu^{r}_{t}\nu^{s}\lambda^{t} - \lambda_{rs}\nu^{r}\nu^{s}_{t}\lambda^{t} .$$

Due to (18a)-1 and (18c)-3, the second term giving $k_{1/3}$ is $\gamma_2\sigma_1$; while due to (19b) and (18a)-3, the third term is $t_1\epsilon_3 + \gamma_1^2$. Due to (18a)-3 and (18c)-1, the second term giving $\gamma_{1/1}$ is $t_1\varepsilon_3$; while due to (19b) and (18c)-1, the third term is $k_1^2+t_1^2$. Next, the indices s and t in the first term giving $\gamma_{1/1}$ are interchanged, and the entire equation is added to that giving $k_{1/3}$. Upon considering the type of (D.5), the first term of the resulting equation is R(1,3,1,3); the other terms are obtained by adding $t_1\varepsilon_3+k_1^2+t_1^2$ to $\gamma_2\sigma_1+t_1\varepsilon_3+\gamma_1^2$ as indicated above. Accordingly, we have

$$k_{1/3} + \gamma_{1/1} = R(1,3,1,3) + k_1^2 + t_1^2 + \gamma_1^2 + 2t_1\varepsilon_3 + \gamma_2\sigma_1 , \qquad (D.7)$$

which is (31b).

From the first alternatives in (15) and (16a), we form

$$t_{1/3} = \lambda_{rst}\nu^r\mu^s\nu^t + \lambda_{rs}\nu^r_t\mu^s\nu^t + \lambda_{rs}\nu^r\mu^s_t\nu^t ,$$

$$\gamma_{1/2} = -\lambda_{rst}\nu^r\nu^s\mu^t - \lambda_{rs}\nu^r_t\nu^s\mu^t - \lambda_{rs}\nu^r\nu^s_t\mu^t .$$

Due to (18a)-2 and (18c)-3, the second term giving $t_{1/3}$ is $\gamma_2\sigma_2$; while due to (19b) and (18b)-3, the third term is $-k_1\varepsilon_3+\gamma_1\gamma_2$. Due to (18a)-3 and (18c)-2, the second term giving $\gamma_{1/2}$ is $k_2\varepsilon_3$; while due to (19b) and (18c)-2, the third term is $k_1t_1+k_2t_1=2Ht_1$, where the statement below equation (30) has been taken into account. Next, the indices s and t in the first term giving $\gamma_{1/2}$ are interchanged, and the entire equation is added to that giving $t_{1/3}$. Upon considering the type of (D.5), the first term of the resulting equation is R(1,3,2,3)=R(2,3,1,3), where the type of (D.4c) has been included; the other terms are obtained by adding $k_2\varepsilon_3+2Ht_1$ to $\gamma_2\sigma_2-k_1\varepsilon_3+\gamma_1\gamma_2$ as indicated above. Accordingly, we have

$$t_{1/3} + \gamma_{1/2} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 - (k_1-k_2)\varepsilon_3 + \gamma_2\sigma_2 , \qquad (D.8)$$

which is (31c).

From the third alternative in (15) and the first alternative in (14b), we form

$$t_{1/2} = \mu_{rst}\nu^r\lambda^s\mu^t + \mu_{rs}\nu^r_t\lambda^s\mu^t + \mu_{rs}\nu^r\lambda^s_t\mu^t ,$$

$$k_{2/1} = \mu_{rst}\nu^r\mu^s\lambda^t + \mu_{rs}\nu^r_t\mu^s\lambda^t + \mu_{rs}\nu^r\mu^s_t\lambda^t .$$

Due to (18b)-1 and (18c)-2, the second term giving $t_{1/2}$ is $t_1\sigma_1$; while due to (19c) and (18a)-2, the third term is $k_2\sigma_2-t_1\gamma_2$. Due to (18b)-2 and (18c)-1, the second term giving $k_{2/1}$ is $k_1\sigma_2$; while due to (19c) and (18b)-1, the third term is $-t_1\sigma_1-t_1\gamma_2$. Next, the indices s and t in the first term giving $k_{2/1}$ are interchanged, and the entire equation is subtracted from that giving $t_{1/2}$. Upon considering the type of (D.5), the first term of the resulting equation is R(2,3,1,2); the other terms are obtained by subtracting $k_1\sigma_2-t_1\sigma_1-t_1\gamma_2$ from $t_1\sigma_1+k_2\sigma_2-t_1\gamma_2$ as indicated above. Accordingly, we have

$$t_{1/2} - k_{2/1} = R(2,3,1,2) - (k_1-k_2)\sigma_2 + 2t_1\sigma_1 , \qquad (D.9)$$

which is (31d).

From the third alternative in (15) and the first alternative in (16b), we form

$$t_{1/3} = \mu_{rst}\nu^r\lambda^s\nu^t + \mu_{rs}\nu^r_t\lambda^s\nu^t + \mu_{rs}\nu^r\lambda^s_t\nu^t ,$$

$$\gamma_{2/1} = -\mu_{rst}\nu^r\nu^s\lambda^t - \mu_{rs}\nu^r_t\nu^s\lambda^t - \mu_{rs}\nu^r\nu^s_t\lambda^t .$$

Due to (18b)-1 and (18c)-3, the second term giving $t_{1/3}$ is $-\gamma_1\sigma_1$; while due to (19c) and (18a)-3, the third term is $k_2\varepsilon_3+\gamma_1\gamma_2$. Due to (18b)-3 and (18c)-1, the second term giving $\gamma_{2/1}$ is $-k_1\varepsilon_3$; while due to (19c) and (18c)-1, the third term is $k_1t_1+k_2t_1=2Ht_1$, where the statement below equation (30) has been taken into account. Next, the indices s and t in the first term giving $\gamma_{2/1}$ are interchanged, and the entire equation is added to that giving $t_{1/3}$. Upon considering the type of (D.5), the first term of the resulting equation is R(2,3,1,3); the other terms are obtained by adding $-k_1\varepsilon_3+2Ht_1$ to $-\gamma_1\sigma_1+k_2\varepsilon_3+\gamma_1\gamma_2$ as indicated above. Accordingly, we have

$$t_{1/3} + \gamma_{2/1} = R(2,3,1,3) + 2Ht_1 + \gamma_1\gamma_2 - (k_1-k_2)\varepsilon_3 - \gamma_1\sigma_1 , \qquad (D.10)$$

which is (31e).

From the first alternatives in (14b) and (16b), we form

$$k_{2/3} = \mu_{rst}\nu^r\mu^s\nu^t + \mu_{rs}\nu^r_t\mu^s\nu^t + \mu_{rs}\nu^r\mu^s_t\nu^t ,$$

$$\gamma_{2/2} = -\mu_{rst}\nu^r\nu^s\mu^t - \mu_{rs}\nu^r_t\nu^s\mu^t - \mu_{rs}\nu^r\nu^s_t\mu^t .$$

Due to (18b)-2 and (18c)-3, the second term giving $k_{2/3}$ is $-\gamma_1\sigma_2$; while due to (19c) and (18b)-3, the third term is $-t_1\varepsilon_3+\gamma_2^2$. Due to (18b)-3 and (18c)-2, the second term giving $\gamma_{2/2}$ is $-t_1\varepsilon_3$; while due to (19c) and (18c)-2, the third term is $t_1^2+k_2^2$. Next, the indices s and t in the first term giving $\gamma_{2/2}$ are interchanged, and the entire equation is added to that giving $k_{2/3}$. Upon considering the type of (D.5), the first term of the resulting equation is R(2,3,2,3); the other terms are obtained by adding $-t_1\varepsilon_3+t_1^2+k_2^2$ to $-\gamma_1\sigma_2-t_1\varepsilon_3+\gamma_2^2$ as indicated above. Accordingly, we have

$$k_{2/3} + \gamma_{2/2} = R(2,3,2,3) + k_2^2 + t_1^2 + \gamma_2^2 - 2t_1\varepsilon_3 - \gamma_1\sigma_2 \, , \qquad (D.11)$$

which is (31f).

From the first alternatives in (13a,b), we form

$$\sigma_{1/2} = \lambda_{rst}\mu^r\lambda^s\mu^t + \lambda_{rs}\mu^r_t\lambda^s\mu^t + \lambda_{rs}\mu^r\lambda^s_t\mu^t \, .$$

$$\sigma_{2/1} = \lambda_{rst}\mu^r\mu^s\lambda^t + \lambda_{rs}\mu^r_t\mu^s\lambda^t + \lambda_{rs}\mu^r\mu^s_t\lambda^t \, .$$

Due to (18a)-1 and (18b)-2, the second term giving $\sigma_{1/2}$ is k_1k_2; while due to (19a) and (18a)-2, the third term is $\sigma_2^2+t_1\varepsilon_3$. Due to (18a)-2 and (18b)-1, the second term giving $\sigma_{2/1}$ is t_1^2; while due to (19a) and (18b)-1, the third term is $-\sigma_1^2+t_1\varepsilon_3$. Next, the indices s and t in the first term giving $\sigma_{2/1}$ are interchanged, and the entire equation is subtracted from that giving $\sigma_{1/2}$. Upon considering the type of (D.5), the first term of the resulting equation is R(1,2,1,2); the other terms are obtained by subtracting $t_1^2-\sigma_1^2+t_1\varepsilon_3$ from $k_1k_2+\sigma_2^2+t_1\varepsilon_3$ as indicated above. Accordingly, we have

$$\sigma_{1/2} - \sigma_{2/1} = R(1,2,1,2) + k_1k_2 - t_1^2 + \sigma_1^2 + \sigma_2^2 \, , \qquad (D.12)$$

which is (31g).

From the first alternatives in (13a) and (17), we form

$$\sigma_{1/3} = \lambda_{rst}\mu^r\lambda^s\nu^t + \lambda_{rs}\mu^r_t\lambda^s\nu^t + \lambda_{rs}\mu^r\lambda^s_t\nu^t \, .$$

$$\varepsilon_{3/1} = \lambda_{rst}\mu^r\nu^s\lambda^t + \lambda_{rs}\mu^r_t\nu^s\lambda^t + \lambda_{rs}\mu^r\nu^s_t\lambda^t \, .$$

Due to (18a)-1 and (18b)-3, the second term giving $\sigma_{1/3}$ is $-k_1\gamma_2$; while due to (19a) and (18a)-3, the third term is $\varepsilon_3\sigma_2-\gamma_1\varepsilon_3$. Due to (18a)-3 and (18b)-1, the second term giving $\varepsilon_{3/1}$ is $-t_1\gamma_1$; while due to (19a) and (18c)-1, the third term is $-k_1\sigma_1-t_1\sigma_2$. Next, the indices s and t in the first term giving $\varepsilon_{3/1}$ are interchanged, and the entire equation is subtracted from that giving $\sigma_{1/3}$. Upon considering the type of (D.5), the first term of the resulting equation is R(1,2,1,3)=R(1,3,1,2), where the type of (D.4c) has been included; the other terms are obtained by subtracting $-t_1\gamma_1-k_1\sigma_1-t_1\sigma_2$ from $-k_1\gamma_2+\varepsilon_3\sigma_2-\gamma_1\varepsilon_3$ as indicated above. Accordingly, we have

$$\sigma_{1/3} - \varepsilon_{3/1} = R(1,3,1,2) - k_1\gamma_2 + t_1\gamma_1 + k_1\sigma_1 + t_1\sigma_2 - \gamma_1\varepsilon_3 + \varepsilon_3\sigma_2 \ . \quad \text{(D.13)}$$

which is (31h).

Finally, from the first alternatives in (13b) and (17), we form

$$\sigma_{2/3} = \lambda_{rst}\mu^r\mu^s\nu^t + \lambda_{rs}\mu^r_t\mu^s\nu^t + \lambda_{rs}\mu^r\mu^s_t\nu^t \ ,$$

$$\varepsilon_{3/2} = \lambda_{rst}\mu^r\nu^s\mu^t + \lambda_{rs}\mu^r_t\nu^s\mu^t + \lambda_{rs}\mu^r\nu^s_t\mu^t \ .$$

Due to (18a)-2 and (18b)-3, the second term giving $\sigma_{2/3}$ is $-t_1\gamma_2$; while due to (19a) and (18b)-3, the third term is $-\varepsilon_3\sigma_1-\gamma_2\varepsilon_3$. Due to (18a)-3 and (18b)-2, the second term giving $\varepsilon_{3/2}$ is $-k_2\gamma_1$; while due to (19a) and (18c)-2, the third term is $-t_1\sigma_1-k_2\sigma_2$. Next, the indices s and t in the first term giving $\varepsilon_{3/2}$ are interchanged, and the entire equation is subtracted from that giving $\sigma_{2/3}$. Upon considering the type of (D.5), the first term of the resulting equation is $R(1,2,2,3)=R(2,3,1,2)$, where the type of (D.4c) has been included; the other terms are obtained by subtracting $-k_2\gamma_1-t_1\sigma_1-k_2\sigma_2$ from $-t_1\gamma_2-\varepsilon_3\sigma_1-\gamma_2\varepsilon_3$ as indicated above. Accordingly, we have

$$\sigma_{2/3} - \varepsilon_{3/2} = R(2,3,1,2) + k_2\gamma_1 - t_1\gamma_2 + t_1\sigma_1 + k_2\sigma_2 - \gamma_2\varepsilon_3 - \varepsilon_3\sigma_1 \ . \quad \text{(D.14)}$$

which is (31i).

D.2 Leg Derivatives of Curvatures Developed Using Surface Vectors

We now re-derive equations (31a,d,g) using the surface vectors λ and μ, thus providing verifications for the formulas developed above. In this task, we make use of the covariant R-tensor in two dimensions. Equations (D.1a-d) remain valid provided the Roman indices are substituted for by the Greek indices (restricted to the numbers 1,2), such as in

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \ , \qquad R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} \ . \quad \text{(D.15a,b)}$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \ , \qquad R_{\alpha\beta\gamma\delta} = R_{\delta\gamma\beta\alpha} \ . \quad \text{(D.15c,d)}$$

As in (5.16) of [H], the covariant R-tensor in a two-dimensional space, i.e., a surface, serves in the definition of K, the Gaussian curvature of the surface:

$$K = (1/4)\varepsilon^{\alpha\beta}\varepsilon^{\gamma\delta}R_{\alpha\beta\gamma\delta} \ .$$

Upon using the first equations of (2.32) in [H], namely

$$\varepsilon^{\alpha\beta} = \lambda^{\alpha}\mu^{\beta} - \mu^{\alpha}\lambda^{\beta} ,$$

and taking advantage of (D.15a,b,d), the Gaussian curvature becomes

$$K = R_{\alpha\beta\gamma\delta}\lambda^{\alpha}\mu^{\beta}\lambda^{\gamma}\mu^{\delta} , \tag{D.16}$$

which is the outcome obtained by a different route in §5-20 of [H]. When working in two dimensions, we do not use the convention of the type (D.3).

Equation (8.31) in [H] yields the Gaussian curvature of a surface embedded in a general space as

$$K = k_1 k_2 - t_1^2 + C , \tag{D.17a}$$

where, according to (5.25) in [H], we have

$$C = R_{urst}\lambda^{u}\mu^{r}\lambda^{s}\mu^{t} \equiv R(1,2,1,2) . \tag{D.17b}$$

Clearly, in the flat space we recover the familiar formula

$$K = k_1 k_2 - t_1^2 . \tag{D.17c}$$

Upon considering (D.16) and (D.17a,b), it follows that

$$R_{\alpha\beta\gamma\delta}\lambda^{\alpha}\mu^{\beta}\lambda^{\gamma}\mu^{\delta} = R(1,2,1,2) + k_1 k_2 - t_1^2 , \tag{D.18}$$

which illustrates why the convention (D.3) cannot be used in conjunction with the Greek indices.

Similar to the three-dimensional case, for the surface one writes according to (5.22) in [H]:

$$\lambda_{\alpha\beta\gamma} - \lambda_{\alpha\gamma\beta} = R^{\delta}_{\ .\alpha\beta\gamma}\lambda_{\delta} = R_{\delta\alpha\beta\gamma}\lambda^{\delta} . \tag{D.19}$$

In reference to (D.2), certain leg derivatives can be obtained also in terms of surface coordinates, as in

$$k_{1/2} = (k_1)_t \mu^t = (k_1)_\gamma \mu^\gamma . \tag{D.20}$$

Further, from (6.07) in [H] we have

$$x^r_\alpha v^\alpha = v^r . \tag{D.21}$$

where v is a surface vector; in the present context it will be either λ or μ. Finally, equation (6.22) in [H], namely

$$b_{\alpha\beta\gamma} - b_{\alpha\gamma\beta} = -R_{urst}\nu^u x^r_\alpha x^s_\beta x^t_\gamma \, , \tag{D.22}$$

will prove useful when contracted three times with a permutation of the surface vectors λ and μ. If we contract (D.22) with $\lambda^\alpha\lambda^\beta\mu^\gamma$, in conjunction with (D.21) we obtain

$$(b_{\alpha\beta\gamma} - b_{\alpha\gamma\beta})\lambda^\alpha\lambda^\beta\mu^\gamma = -R(3,1,1,2) = R(1,3,1,2) \, , \tag{D.23a}$$

while if we contract it with $\mu^\alpha\lambda^\beta\mu^\gamma$, we obtain

$$(b_{\alpha\beta\gamma} - b_{\alpha\gamma\beta})\mu^\alpha\lambda^\beta\mu^\gamma = -R(3,2,1,2) = R(2,3,1,2) \, . \tag{D.23b}$$

The initial equations serving for the derivation of (31a,d,g) in two dimensions are (13a')-(15'). The development involves (D.18,19) and (D.23a,b) above, as well as (13a')-(15') and (18a',b'), where (19a') could be used instead of (13a',b'). Keeping in mind surface expressions of the type (D.20), from (14a') and the first alternative in (15'), we form

$$k_{1/2} = b_{\alpha\beta\gamma}\lambda^\alpha\lambda^\beta\mu^\gamma + b_{\alpha\beta}\lambda^\alpha_\gamma\lambda^\beta\mu^\gamma + b_{\alpha\beta}\lambda^\alpha\lambda^\beta_\gamma\mu^\gamma \, ,$$

$$t_{1/1} = b_{\alpha\beta\gamma}\lambda^\alpha\mu^\beta\lambda^\gamma + b_{\alpha\beta}\lambda^\alpha_\gamma\mu^\beta\lambda^\gamma + b_{\alpha\beta}\lambda^\alpha\mu^\beta_\gamma\lambda^\gamma \, .$$

Due to the second formula in (18a'), symbolized by (18a')-2, and to the second alternative in (15'), the second term giving $k_{1/2}$ is $t_1\sigma_2$; while due to (18a')-2 and the first alternative in (15'), the third term is again $t_1\sigma_2$. Due to (18a')-1 and (14b'), the second term giving $t_{1/1}$ is $k_2\sigma_1$; while due to (18b')-1 and (14a'), the third term is $-k_1\sigma_1$. Next, the indices β and γ in the first term giving $t_{1/1}$ are interchanged, and the entire equation is subtracted from that giving $k_{1/2}$. Upon considering (D.23a), the first term of the resulting equation is R(1,3,1,2); the other terms are obtained by subtracting $k_2\sigma_1 - k_1\sigma_1$ from $2t_1\sigma_2$ as indicated above. Accordingly, we have

$$k_{1/2} - t_{1/1} = R(1,3,1,2) + (k_1 - k_2)\sigma_1 + 2t_1\sigma_2 \, , \tag{D.24}$$

the same as (D.6), thereby confirming (31a). We note that the identity of (D.6) and (D.24) emerges in the last step of two different and independent approaches.

Similarly, from the second alternative in (15') and from (14b'), we form

$$t_{1/2} = b_{\alpha\beta\gamma}\mu^\alpha\lambda^\beta\mu^\gamma + b_{\alpha\beta}\mu^\alpha_\gamma\lambda^\beta\mu^\gamma + b_{\alpha\beta}\mu^\alpha\lambda^\beta_\gamma\mu^\gamma ,$$

$$k_{2/1} = b_{\alpha\beta\gamma}\mu^\alpha\mu^\beta\lambda^\gamma + b_{\alpha\beta}\mu^\alpha_\gamma\mu^\beta\lambda^\gamma + b_{\alpha\beta}\mu^\alpha\mu^\beta_\gamma\lambda^\gamma .$$

Due to (18b')-2 and (14a'), the second term giving $t_{1/2}$ is $-k_1\sigma_2$; while due to (18a')-2 and (14b'), the third term is $k_2\sigma_2$. Due to (18b')-1 and the first alternative in (15'), the second term giving $k_{2/1}$ is $-t_1\sigma_1$; while due to (18b')-1 and the second alternative in (15'), the third term is again $-t_1\sigma_1$. Next, the indices β and γ in the first term giving $k_{2/1}$ are interchanged, and the entire equation is subtracted from that giving $t_{1/2}$. Upon considering (D.23b), the first term of the resulting equation is R(2,3,1,2); the other terms are obtained by subtracting $-2t_1\sigma_1$ from $-k_1\sigma_2+k_2\sigma_2$ as indicated above. Accordingly, we have

$$t_{1/2} - k_{2/1} = R(2,3,1,2) - (k_1-k_2)\sigma_2 + 2t_1\sigma_1 , \qquad (D.25)$$

which is the same outcome as (D.9). Thus, (31d) is confirmed by different and independent means.

Finally, from the first alternatives in (13a',b'), we form

$$\sigma_{1/2} = \lambda_{\alpha\beta\gamma}\mu^\alpha\lambda^\beta\mu^\gamma + \lambda_{\alpha\beta}\mu^\alpha_\gamma\lambda^\beta\mu^\gamma + \lambda_{\alpha\beta}\mu^\alpha\lambda^\beta_\gamma\mu^\gamma ,$$

$$\sigma_{2/1} = \lambda_{\alpha\beta\gamma}\mu^\alpha\mu^\beta\lambda^\gamma + \lambda_{\alpha\beta}\mu^\alpha_\gamma\mu^\beta\lambda^\gamma + \lambda_{\alpha\beta}\mu^\alpha\mu^\beta_\gamma\lambda^\gamma .$$

Due to (18b')-2 and $\lambda_{\alpha\beta}\lambda^\alpha=0$ (upon considering Hotine's equation 3.19 in two dimensions), the second term giving $\sigma_{1/2}$ is 0; while due to (18a')-2 and the first alternative in (13b'), the third term is σ_2^2. Due to (18b')-1 and $\lambda_{\alpha\beta}\lambda^\alpha=0$, the second term giving $\sigma_{2/1}$ is 0; while due to (18b')-1 and the first alternative in (13a'), the third term is $-\sigma_1^2$. Both these third terms can equivalently be obtained if (19a') is used instead of (13a',b'). Next, the indices β and γ in the first term giving $\sigma_{2/1}$ are interchanged, and the entire equation is subtracted from that giving $\sigma_{1/2}$. Upon considering (D.19) and (D.18), the first term of the resulting equation is $R(1,2,1,2)+k_1k_2-t_1^2$; the other terms are obtained by subtracting $-\sigma_1^2$ from σ_2^2 as indicated above. Accordingly, we have

$$\sigma_{1/2} - \sigma_{2/1} = R(1,2,1,2) + k_1k_2 - t_1^2 + \sigma_1^2 + \sigma_2^2 , \qquad (D.26)$$

which is the same outcome as (D.12), confirming (31g) by independent means.

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